SEMI-CLASSICAL GREEN KERNEL ASYMPTOTICS FOR THE DIRAC OPERATOR

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ABSTRACT. We consider a semi-classical Dirac operator in $d \in \mathbb{N}$ spatial dimensions with a smooth potential whose partial derivatives of any order are bounded by suitable constants. We prove that the distribution kernel of the inverse operator evaluated at two distinct points fulfilling a certain hypothesis can be represented as the product of an exponentially decaying factor involving an associated Agmon distance and some amplitude admitting a complete asymptotic expansion in powers of the semi-classical parameter. Moreover, we find an explicit formula for the leading term in that expansion.

1. Introduction and main results

The free Dirac operator in $d \in \mathbb{N}$ spatial dimensions is the matrix-valued partial differential operator given by

$$D_{h,0} := \boldsymbol{\alpha} \cdot (-ih\nabla) + \alpha_0 := \sum_{k=1}^{d} \alpha_k \, (-ih\,\partial_{x_k}) + \alpha_0 \,, \qquad h \in (0,1] \,. \tag{1.1}$$

The Dirac matrices $\alpha_0, \ldots, \alpha_d$ appearing here are hermitian $(d_* \times d_*)$-matrices satisfying the Clifford algebra relations

$$\{\alpha_k \,,\, \alpha_\ell\} = 2\,\delta_{k\ell}\,\mathbb{1} \,, \qquad k, \ell = 0, 1, \ldots, d \,. \tag{1.2}$$

According to the representation theory of Clifford algebras such matrices exist and the minimal choice of their dimension $d_* \in 2\mathbb{N}$ is $d_* = 2^{[(d+1)/2]}$. The special choice of the Dirac matrices is immaterial for our purposes; only the relations (1.2) are used explicitly below. It is well-known that, as an operator acting in the Hilbert space $L^2(\mathbb{R}^d, \mathbb{C}^{d_*})$, $D_{h,0}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d, \mathbb{C}^{d_*})$ and self-adjoint on $H^1(\mathbb{R}^d, \mathbb{C}^{d_*})$. Its Fourier transform can be easily diagonalized (compare (2.6) and (2.7) below) revealing that its spectrum is purely absolutely continuous and given as

$$\sigma(D_{h,0}) = \sigma_{\mathrm{ac}}(D_{h,0}) = (-\infty, -1] \cup [1, \infty) \,. \tag{1.3}$$

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Next, we add a smooth potential, V, to the free Dirac operator,

$$D_{h,V} := D_{h,0} + V\,\mathbb{1}_{d_*}\,. \tag{1.4}$$

We shall always assume that V has the following properties.

Hypothesis 1.1. $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ *and, for every multi-index* $\alpha \in \mathbb{N}_0^d$,

$$\sup_{x\in\mathbb{R}^d} |\partial_x^\alpha V(x)| < \infty\,. \tag{1.5}$$

Moreover, there is some $\delta \in (0,1)$ *such that*

$$-1+\delta \leqslant V(x) \leqslant -\delta\,, \qquad x \in \mathbb{R}^d. \tag{1.6}$$

In view of (1.3) the previous hypothesis clearly implies that $D_{h,V}$ is self-adjoint on $H^1(\mathbb{R}^d, \mathbb{C}^{d_*})$ and continuously invertible. In fact, its symbol,

$$\widehat{D}_V(x,\xi) := \boldsymbol{\alpha}\cdot\xi + \alpha_0 + V(x)\,, \qquad (x,\xi) \in \mathbb{R}^{2d},$$

is uniformly elliptic in the sense that

$$\big|\det\big(\widehat{D}_V(x,\xi)\big)\big| = \big(1+|\xi|^2 - V^2(x)\big)^{d_*/2} \geqslant (2\delta-\delta^2)^{d_*/2} > 0\,, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

Therefore, the inverse $D_{h,V}^{-1}$ is given by some matrix-valued h-pseudo-differential operator whose distribution kernel, $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto D_{h,V}^{-1}(x,y)$, is smooth away from the diagonal. Our goal is to study the semi-classical asymptotics of this kernel, for fixed $x \neq y$.

To formulate our main result we first introduce an associated Agmon distance, d_{A}, on $\mathbb{R}^d$. It is the Riemannian distance corresponding to a metric conformally equivalent to the Euclidean one on $\mathbb{R}^d$, namely

$$G(x) := (1-V^2(x))\,\mathbb{1}_d\,, \qquad x \in \mathbb{R}^d. \tag{1.7}$$

The Agmon distance is thus given as

$$d_{\mathrm{A}}(x,y) := \inf_{q:y\rightsquigarrow x} \int \big\langle\, \dot q \,\big|\, G(q)\,\dot q\,\big\rangle^{1/2}, \qquad x,y \in \mathbb{R}^d, \tag{1.8}$$

where the infimum is taken over all piecewise smooth paths $q:[0,b]\to\mathbb{R}^d$, for some $b>0$, such that $q(0)=y$ and $q(b)=x$. We also introduce an associated Hamilton function,

$$H(x,p) := -\sqrt{1-|p|^2} - V(x)\,, \qquad x,p \in \mathbb{R}^d\,,\ |p|<1\,, \tag{1.9}$$

and recall the following fact (see, e.g., [2, pp. 197]): If a smooth curve $\binom{\gamma}{\varpi}: I \to \mathbb{R}^{2d}$ on a non-trivial interval I is a solution of the Hamiltonian equations

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \nabla_p H \\ -\nabla_x H \end{pmatrix}(x,p) \tag{1.10}$$

such that

$$H(\gamma(t),\varpi(t)) = 0\,, \qquad t \in I\,, \tag{1.11}$$

then γ is a geodesic for the Agmon metric G. Let $\exp_y : T_y\mathbb{R}^d \to \mathbb{R}^d$ denote the exponential map at $y \in \mathbb{R}^d$ associated to the Riemannian metric G. We recall that two points $x, y \in \mathbb{R}^d$ are called conjugate to each other iff the derivative $\exp_y'(v)$ is singular, where $v \in \mathbb{R}^d$ is chosen such that $\exp_y(v) = x$. In this article we shall restrict our attention to arguments of the Green kernel fulfilling the following hypothesis.

Hypothesis 1.2. *$x_\star, y_\star \in \mathbb{R}^d$, $x_\star \neq y_\star$, and, up to reparametrization, there is a unique minimizing geodesic from $y_\star$ to $x_\star$. Moreover, $x_\star$ and $y_\star$ are not conjugate to each other.*

We recall that Hypothesis 1.2 is always fulfilled, for fixed $y_\star$, provided that $x_\star$ is sufficiently close to $y_\star$. To state our main results we also introduce the – in general non-orthogonal – projections $\Lambda^\pm$ defined by

$$\Lambda^\pm(\zeta) := \frac{1}{2}\mathbb{1} \pm \frac{1}{2}S(\zeta)\,, \quad S(\zeta) := \frac{\boldsymbol{\alpha}\cdot\zeta + \alpha_0}{\sqrt{1+\zeta^2}}\,, \quad \zeta \in \mathbb{C}^d,\ |\Im\zeta| < 1\,; \tag{1.12}$$

compare Subsection 2.1. Here and henceforth we abbreviate $\zeta^2 := \zeta_1^2 + \cdots + \zeta_d^2$, for every $\zeta \in \mathbb{C}^d$, and $\sqrt{\cdot}$ denotes the branch of the square root slit on the negative real axis satisfying $\Re\sqrt{\cdot} > 0$. The following theorem presents the main result of this article in the case $d \geqslant 2$.

Theorem 1.3. *Let $d \geqslant 2$ and assume that V fulfills Hypothesis 1.1 and $x_\star, y_\star$ fulfill Hypothesis 1.2. Let $\binom{\gamma}{\varpi} : [0,\tau] \to \mathbb{R}^{2d}$ be a smooth curve solving* (1.10) *and satisfying* (1.11) *such that $\gamma(0) = y_\star$ and $\gamma(\tau) = x_\star$. Then, as $h > 0$ tends to zero,*

$$\begin{aligned} D_{h,V}^{-1}(x_\star, y_\star) = \frac{1}{h^d} \cdot \frac{(1-V^2(x_\star))^{\frac{d-2}{4}}(1-V^2(y_\star))^{\frac{d-2}{4}}}{\det\big[\exp_{y_\star}'(\exp_{y_\star}^{-1}(x_\star))\big]^{1/2}} \cdot \frac{(1+\mathcal{O}(h))\,e^{-d_{\mathrm{A}}(x_\star,y_\star)/h}}{\big(2\pi\, d_{\mathrm{A}}(x_\star,y_\star)/h\big)^{\frac{d-1}{2}}} \\ \cdot\, U(\tau)\,(-V(y_\star))\Lambda^+(i\varpi(0))\,, \end{aligned} \tag{1.13}$$

where $U(t)$, $t \in [0,\tau]$, is a unitary matrix such that U solves the matrix-valued initial value problem

$$\frac{d}{dt}U(t) \;=\; -\frac{i\boldsymbol{\alpha}}{2}\cdot\frac{\nabla V(\gamma(t))}{V(\gamma(t))}\,U(t)\,,\ t \in [0,\tau]\,, \qquad U(0) = \mathbb{1}\,.$$

The term abbreviated by $\mathcal{O}(h)$ in (1.13) *admits a complete asymptotic expansion in powers of h.*

Proof. This theorem follows from (2.1), (2.2), Proposition 6.1, and Lemma 6.4 below. □

Remark 1.4. (i) The factor $\varrho(x,y) := \det\big[\exp_y'(\exp_y^{-1}(x))\big]^{1/2}$ is familiar from the asymptotic expansion of the heat kernel associated to G, where it also

appears in the denominator of the leading coefficient. In particular, is it known to be symmetric, $\varrho(x,y) = \varrho(y,x)$.

(ii) It follows from Remark 4.7 that $M(x_\star, y_\star) := U(\tau)\,(-V(y_\star))\Lambda^+(i\varpi(0)) = (-V(x_\star))\Lambda^+(i\varpi(\tau))\,\alpha_0\,U(\tau)\,(-V(y_\star))\Lambda^+(i\varpi(0))$. Using the latter formula we verify in the same remark that $M(x_\star, y_\star)^* = M(y_\star, x_\star)$ so that (1.13) has the correct symmetry property of the kernel of a matrix-valued self-adjoint operator.

(iii) In Appendix A we explain, in the case $d = 3$, the connection between the term $U(\tau)\,(-V(y_\star))\Lambda^+(i\varpi(0))$ and the BMT equation for the Thomas precession of a classical spin along a particle trajectory. The BMT equation is discussed in connection with the semi-classical analysis of the time evolution generated by $D_{h,V}$ in [1, 8]. $\diamond$

Next, we state our main result in the case $d = 1$, where we do not need any restriction on the entries $x \neq y$ of the Green kernel.

Theorem 1.5. *Let $x, y \in \mathbb{R}$, $x \neq y$, and assume that V fulfills Hypothesis 1.1 with $d = 1$. Let $\binom{\gamma}{\varpi} : [0,\tau] \to \mathbb{R}^{2d}$ be a smooth curve solving* (1.10) *and satisfying* (1.11) *such that $\gamma(0) = y$ and $\gamma(\tau) = x$. Then, as $h > 0$ tends to zero,*

$$\begin{aligned} D_{h,V}^{-1}(x,y) = \frac{1}{h} \cdot \frac{(1 + \mathcal{O}(h))\,\exp\Big(-\Big|\int_y^x \big(1 - V^2(t)\big)^{1/2}\,dt\Big|\Big/h\Big)}{(1 - V^2(x))^{1/4}(1 - V^2(y))^{1/4}} \\ \cdot \big(\cos(\vartheta(\tau))\,\mathbb{1} - i\sin(\vartheta(\tau))\,\alpha_1\big)\,(-V(y))\,\Lambda^+(i\omega(0))\,, \end{aligned} \tag{1.14}$$

where

$$\vartheta(\tau) := \int_0^\tau \frac{V'(\gamma(t))}{2V(\gamma(t))}\,dt\,.$$

The term abbreviated by $\mathcal{O}(h)$ in (1.14) *admits a complete asymptotic expansion in powers of h.*

Proof. This theorem follows from (2.1), (2.2), and Proposition 6.2. $\square$

Remark 1.6. The choice of the sign of V in Hypothesis 1.1 is not important for our results. We could equally well consider smooth potentials $V : \mathbb{R}^d \to \mathbb{R}$ satisfying (1.5) and $0 < \delta \leqslant V \leqslant 1 - \delta$. In fact, this immediately follows from the following trivial observation: If $\alpha_0, \dots, \alpha_d$ are Dirac matrices and $D_{h,V}$ is defined as in (1.1) and (1.4) with some positive V, then $D_{h,V} = -\widetilde{D}_{h,-V}$, where $\widetilde{D}_{h,-V} := \widetilde{\boldsymbol{\alpha}} \cdot (-ih\nabla) + \widetilde{\alpha}_0 - V$. Here the new Dirac matrices $\widetilde{\alpha}_j := -\alpha_j$, $j = 0, \dots, d$, again satisfy (1.2) and, hence, Theorems 1.3 and 1.5 are applicable to $\widetilde{D}_{h,-V}$. In doing so we first observe that, for two given points $x_\star, y_\star$, the validity of Hypothesis 1.2 does not depend on the sign of V since the Agmon metric $G = (1 - V^2)\,\mathbb{1}$ depends only on V^2. Moreover, the expression in the

first line of the right hand side of (1.13), which we denote by $\Delta(x_\star, y_\star)$, does not depend on the sign of V either, since the Agmon distance and the exponential map are defined by means of G. We have, however, to introduce a new Hamilton function,

$$\widetilde{H}(x,p) := -\sqrt{1-|p|^2} + V(x)\,, \qquad x \in \mathbb{R}^d,\ |p| < 1\,.$$

Let $\binom{\widetilde{\gamma}}{\widetilde{\varpi}} : [0, \widetilde{\tau}]$ be a Hamiltonian trajectory solving (1.10) and (1.11) with H replaced by $\widetilde{H}$ such that $\widetilde{\gamma}(0) = y_\star$ and $\widetilde{\gamma}(\widetilde{\tau}) = x_\star$. Then the last line of (1.13) has to be changed as follows. Since $D_{h,V}^{-1} = -\widetilde{D}_{h,-V}^{-1}$ we obtain

$$D_{h,V}^{-1}(x_\star, y_\star) = -\Delta(x_\star, y_\star)\, \widetilde{U}(\widetilde{\tau})\, V(y_\star) \Lambda^-(i\widetilde{\varpi}(0))\,, \tag{1.15}$$

where the projection Λ^- is again defined with the original α_j and $\widetilde{U}(t)$, $t \in [0, \widetilde{\tau}]$, is a unitary matrix such that $\widetilde{U}$ solves the matrix-valued initial value problem

$$\frac{d}{dt}\widetilde{U}(t) = \frac{i\boldsymbol{\alpha}}{2} \cdot \frac{\nabla V(\widetilde{\gamma}(t))}{V(\widetilde{\gamma}(t))}\, \widetilde{U}(t)\,,\ t \in [0, \widetilde{\tau}]\,, \qquad \widetilde{U}(0) = \mathbb{1}\,.$$

By Remark 1.4(ii) we may multiply the right hand side of (1.15) from the left with $V(x_\star)\, \Lambda^-(i\widetilde{\varpi}(\widetilde{\tau}))\, (-\alpha_0)$. Similar replacements have to be made in Formula (1.14) for the one-dimensional case. ◇

Example 1.7. Assume that $V = E$ is some constant spectral parameter $E \in (-1, 1)$. Then we can compute the Green kernel of $D_{h,E} = \boldsymbol{\alpha} \cdot (-ih\nabla) + \alpha_0 + E$ by means of the Fourier transform and find the well-known expression

$$\begin{aligned} D_{h,E}^{-1}(x,y) &= \frac{D_{h,-E}}{(2\pi)^{d/2}\, h^d} \Big(\frac{|x-y|}{h\sqrt{1-E^2}}\Big)^{1-d/2} K_{d/2}\big(\sqrt{1-E^2}\,|x-y|/h\big) \\ &= \frac{(1-E^2)^{d/4}}{(2\pi)^{d/2}\, h^d} \Big(\frac{r}{h}\Big)^{1-d/2} \Big\{ -i\frac{\boldsymbol{\alpha}\cdot\mathbf{r}}{r}\, K'_{d/2}\big(\sqrt{1-E^2}\, r/h\big) \\ &\quad + \Big(\alpha_0 - E + ih\,(d/2-1)\frac{\boldsymbol{\alpha}\cdot\mathbf{r}}{r^2}\Big) \frac{K_{d/2}\big(\sqrt{1-E^2}\, r/h\big)}{\sqrt{1-E^2}} \Big\}, \end{aligned} \tag{1.16}$$

where we abbreviate $r := |x-y|$ and $\mathbf{r} := x - y$ in the second line. For large ρ, the Bessel function of the second kind, $K_{d/2}$, behaves asymptotically as $K_{d/2}(\rho) = (\pi/2\rho)^{1/2}\, e^{-\rho}\,(1 + \mathcal{O}(1/\rho))$ and $K'_{d/2}(\rho) = -(\pi/2\rho)^{1/2}\, e^{-\rho}\,(1 + \mathcal{O}(1/\rho))$. Moreover, it is clear that $\sqrt{1-E^2}\,\frac{x-y}{|x-y|}$ is the constant momentum of the Hamiltonian trajectory running from y to x in the level set $\{p^2 = 1 - E^2\}$ and we readily verify that $d_{\mathrm{A}}(x,y) = \sqrt{1-E^2}\,|x-y|$, $(1-V^2(x))^{1/4}(1-V^2(y))^{1/4} = \sqrt{1-E^2}$, $\exp'_y = \mathbb{1}$, and $U = \mathbb{1}$. Consequently, the leading asymptotics in (1.16) agrees with the value predicted by Theorems 1.3 and 1.5. ◇

An asymptotic expansion analogous to (1.13) has been derived earlier in [5] for a certain class of h-pseudo-differential operators whose symbols are periodic in

the momentum variables. In the general case encountered in [5] the Agmon metric is replaced by a suitable Finsler metric. This is due to the fact that the figuratrix at $x \in \mathbb{R}^d$,

$$\mathbb{f}_x := \{ p \in \mathbb{R}^d : H(x,p) = 0 \}, \tag{1.17}$$

which is well-defined due to (1.6) and just a sphere with radius $\sqrt{1 - V^2(x)}$, is replaced by the boundary of some more general strictly convex body in more general situations. We also remark that the exponential decay of eigenfunctions and the semi-classical tunneling effect for the Dirac operator is studied by means of the Agmon metric in [10].

We briefly outline the strategy of our proofs and the organization of this article. The first step is to conjugate the Dirac operator with exponential weights $e^{\varphi/h}$ where φ is essentially given as the Agmon distance to $y_\star$; compare (2.1)–(2.3) below. The choice of φ is explained more precisely in Section 2. Its construction is the same as in [5] and we shall not repeat the details of the proofs in Section 2. The distribution kernel of a parametrix of the conjugated Dirac operator yields the prefactor in front of the exponential in (1.13) and (1.14). The symbol of the conjugated Dirac operator, whose properties are also discussed in Section 2, is given by a non-hermitian matrix having two different $(d_*/2)$-fold degenerate eigenvalues, one with a non-negative real part and another one with a strictly negative real part. Only the part corresponding to the eigenvalue with non-negative real part contributes to the asymptotics of the distribution kernel. To obtain the asymptotics we first construct, roughly speaking, a parametrix for a "heat equation" (backwards in time for the part of the symbol belonging to the eigenvalue in the left complex half-plane) by means of a WKB construction. Since the eigenvalues are complex we use a Fourier integral operator with complex-valued phase function as an ansatz for the parametrix and work with almost analytic extensions. In order to solve the associated complex time dependent Hamilton-Jacobi equation we adapt the constructions of [4, 7]. In Section 3 we provide a self-contained discussion of the time dependent Hamilton-Jacobi equation that proceeds along the lines of [7] and provides some alternative arguments to control the derivatives of certain error terms and implicit functions. To solve the transport equations in our WKB construction we employ a strategy based on the Clifford algebra structure we learned from [11]. This strategy is adapted to our setting in Section 4. (WKB constructions for the usual Dirac equation which also apply to non-scalar potentials can be found in [3, 8].) In Section 5 we construct a parametrix for the conjugated Dirac operator by integrating the parametrix for the "heat equation" with respect to the time variable and adding a term accounting for the part of its symbol left out in the WKB construction. Finally, in Section 6 we compute the asymptotics of $e^{\varphi(x)/h}\, D_{h,V}^{-1}(x,y)\, e^{-\varphi(y)/h}$ by means of a stationary phase expansion in the time variable and the momentum variables of the Fourier

integral operator. The main text is followed by an appendix where the BMT equation for Thomas precession is related to our results.

2. Construction of a weight function

2.1. Eigenvalues and eigenprojections of the symbol of the conjugated Dirac operator. Let us describe the first step in the derivation of the asymptotics (1.13) and (1.14). We fix two distinct points, $x_\star$ and $y_\star$, in $\mathbb{R}^d$ fulfilling Hypothesis 1.2 and seek for some bounded weight function $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ satisfying

$$\varphi(x_\star) - \varphi(y_\star) \,=\, d_{\mathrm{A}}(x_\star, y_\star)\,. \tag{2.1}$$

Since φ is bounded and smooth it is then clear that

$$D^{-1}_{h,V}(x_\star, y_\star) \,=\, e^{-\varphi(x_\star)/h}\, D^{-1}_{h,V,\varphi}(x_\star, y_\star)\, e^{\varphi(y_\star)/h}, \tag{2.2}$$

where $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto D^{-1}_{h,V,\varphi}(x,y)$ denotes the distribution kernel of the inverse of the conjugated Dirac operator

$$\begin{aligned} D_{h,V,\varphi} \,&:=\, e^{\varphi/h}\, D_{h,V}\, e^{-\varphi/h} \\ &=\, \boldsymbol{\alpha}\cdot(-ih\,\nabla + i\nabla\varphi) \,+\, \alpha_0 \,+\, V\,\mathbb{1}_{d_*}\,. \end{aligned} \tag{2.3}$$

The Green kernel asymptotics of $D_{h,V,\varphi}$ thus yield the prefactor in front of the exponential in (1.13) and (1.14) provided that φ is chosen in the right way. To motivate the partial differential equation determining φ we observe that the – in general non-hermitian – matrix

$$\widehat{D}_V(x,\zeta) \,:=\, \boldsymbol{\alpha}\cdot\zeta \,+\, \alpha_0 \,+\, V(x)\,\mathbb{1}\,, \qquad (x,\zeta) \in \mathbb{R}^d \times \mathbb{C}^d, \tag{2.4}$$

has two $(d_*/2)$-fold degenerate complex eigenvalues, namely

$$\lambda_\pm(x,\zeta) \,:=\, \pm\sqrt{1+\zeta^2} \,+\, V(x)\,, \qquad (x,\zeta) \in \mathbb{R}^d \times \mathbb{C}^d,\ |\Im\zeta| < 1\,. \tag{2.5}$$

The eigenprojections corresponding to the eigenvalues in (2.5) are given by (1.12). In fact, a straightforward exercise using (1.2), which implies $(\boldsymbol{\alpha}\cdot\zeta)^2 = \zeta^2\,\mathbb{1}$ and, hence, $(\boldsymbol{\alpha}\cdot\zeta + \alpha_0)^2 = (\zeta^2+1)\,\mathbb{1}$ reveals that, for $\zeta \in \mathbb{C}^d$, $|\Im\zeta| < 1$,

$$\Lambda^+(\zeta) + \Lambda^-(\zeta) \,=\, \mathbb{1}\,, \qquad S(\zeta)^2 \,=\, \mathbb{1}\,, \qquad \Lambda^\pm(\zeta)^2 \,=\, \Lambda^\pm(\zeta)\,, \tag{2.6}$$

$$\widehat{D}_V(x,\zeta)\,\Lambda^\pm(\zeta) \,=\, \lambda_\pm(\zeta)\,\Lambda^\pm(\zeta)\,. \tag{2.7}$$

Using $\cos^2(\theta/2) = (1+\cos(\theta))/2 \geqslant \cos(\theta)$, $\theta \in (-\pi/2, \pi/2)$, we further observe for later reference that

$$\Re\sqrt{z} = \sqrt{|z|}\,\cos(\theta/2) \geqslant \sqrt{|z|\,\cos(\theta)} = \sqrt{\Re z}\,, \quad z = |z|\,e^{i\theta} \in \mathbb{C}\,,\ \Re z > 0\,.$$

In particular,

$$\Re\sqrt{1+\zeta^2} \,\geqslant\, \sqrt{1 + (\Re\zeta)^2 - (\Im\zeta)^2}\,, \qquad \zeta \in \mathbb{C}^d\,,\ |\Im\zeta| < 1\,. \tag{2.8}$$

2.2. **Agmon's distance as an optimal weight function.** In this subsection we treat the partial differential equation determining the weight function φ. This equation is the eikonal equation corresponding to the Hamilton function introduced in (1.9) which is related to the eigenvalue λ_+ as

$$H(x,p) \,=\, -\sqrt{1-p^2} - V(x) \,=\, -\lambda_+(x,ip)\,, \qquad x\in\mathbb{R}^d\,,\ |p|<1\,. \tag{2.9}$$

We start with an elementary proposition covering the one-dimensional case.

Proposition 2.1. *Let $d=1$ and assume that V fulfills Hypothesis 1.1. Then the following assertions hold true:*
(i) For $y\in\mathbb{R}$, the unique solution in $C^1(\mathbb{R},\mathbb{R})$ of the initial value problem

$$H(x,\phi'(x)) \,=\, -\sqrt{1-\phi'(x)^2} - V(x) \,=\, 0\,, \quad x\in\mathbb{R}\,, \quad \pm\phi'>0\,, \quad \phi(y)=0\,,$$

is given by the smooth function

$$\phi(x) \,=\, \pm\int_y^x \sqrt{1-V^2(t)}\,dt\,, \qquad x\in\mathbb{R}\,.$$

We have $\phi(x)=\pm d_{\mathrm{A}}(x,y)$, for $x\geqslant y$, and $\phi(x)=\mp d_{\mathrm{A}}(x,y)$, for $x<y$.
(ii) Given $x_\star,y_\star\in\mathbb{R}$, $x_\star\neq y_\star$, we find some compact interval, $K_0\subset\mathbb{R}$, such that $x_\star,y_\star\in\mathring{K}_0$ and some $\varphi\in C^\infty(\mathbb{R},\mathbb{R})$ such that $\varphi(x)-\varphi(y)=d_{\mathrm{A}}(x,y)$, for all $x,y\in K_0$ with $\mathrm{sgn}(x-y)=\mathrm{sgn}(x_\star-y_\star)$, φ is constant near $\pm\infty$, and

$$H(x,\varphi'(x))\leqslant 0\,, \quad x\in\mathbb{R}\,, \quad \textit{and} \quad H(x,\varphi'(x))=0 \ \Leftrightarrow\ x\in K_0\,. \tag{2.10}$$

Proof. (i): Since $-1+\delta\leqslant V\leqslant-\delta$, every solution $\phi\in C^1(\mathbb{R},\mathbb{R})$ of $H(x,\phi'(x))=0$, $x\in\mathbb{R}$, satisfies either $\phi'>0$ or $\phi'<0$ on $\mathbb{R}$. Thus, $H(x,\phi')=0$ is equivalent to either $\phi'=\sqrt{1-V^2}$ or $\phi'=-\sqrt{1-V^2}$ and the first assertion is evident. Since the expression $\int\langle\,\dot q\,|\,G(q)\,\dot q\,\rangle^{1/2}$ is invariant with respect to reparametrizations of the path q, we may plug in $q(t)=y+t$, $t\in[0,x-y]$, for $y\leqslant x$, or $q(t)=y-t$, $t\in[0,y-x]$, for $y>x$, to verify that $\phi(x)=\pm\mathrm{sgn}(x-y)\,d_{\mathrm{A}}(x,y)$.

(ii): We choose some compact interval K_0 with $\mathring{K}_0\ni x_\star,y_\star$ and pick some $\theta\in C_0^\infty(\mathbb{R},[0,1])$ such that $\theta(t)=1$ if and only if $t\in K_0$. Then we define $\varphi(x):=\pm\int_{y_\star}^x\theta(t)\sqrt{1-V^2(t)}\,dt$, $x\in K_0$, where we choose the $+$-sign if and only if $x_\star>y_\star$. By Part (i) φ satisfies $H(x,\varphi')=0$ on K_0 and, for $x\notin K_0$, we have $\varphi'(x)^2=\theta(x)^2(1-V^2(x))<1-V^2(x)$, that is, $H(x,\varphi'(x))<0$. □

To discuss the multi-dimensional case we denote the flow of the Hamiltonian vector field corresponding to H by $\Phi=(X,P):\mathcal{D}(\Phi)\to\mathbb{R}^{2d}$, so that $\mathcal{D}(\Phi)=\{(t,x,p)\in\mathbb{R}\times\mathbb{R}^d\times B_1\,:\,t\in I_{\max}(x,p)\}$ and

$$\partial_t\Phi \,=\, \partial_t\begin{pmatrix}X\\P\end{pmatrix} \,=\, \begin{pmatrix}\nabla_pH(X,P)\\-\nabla_xH(X,P)\end{pmatrix} \qquad \text{on } \mathcal{D}(\Phi)\,. \tag{2.11}$$

Here $I_{\max}(x,p)$ denotes the maximal interval of existence for the initial value problem $\dot\rho = (\nabla_p H(\rho), -\nabla_x H(\rho))$, $\rho(0) = (x,p)$, and $B_1 = \{p \in \mathbb{R}^d : p^2 < 1\}$.

In order to recall a result from [5] we remark that, if Hypothesis 1.2 is satisfied, there is, up to reparametrization, again only one minimizing geodesic running in the opposite direction from $x_\star$ to $y_\star$. Moreover, we can prolong the geodesic from $y_\star$ to $x_\star$ or from $x_\star$ to $y_\star$ a little bit such that it still remains minimizing. Now, we are prepared to recall the following special case of [5, Proposition 4.5]:

Proposition 2.2. *Let $d \geqslant 2$ and assume that V fulfills Hypothesis 1.1 and $x_\star$ and $y_\star$ fulfill Hypothesis 1.2. Then there exist a point, y_0, on the prolongation of the geodesic from $x_\star$ to $y_\star$, a compact neighborhood, K_0, of the geodesic segment from $y_\star$ to $x_\star$, some open set, $\mathscr{W} \subset T_{y_0}\mathbb{R}^d = \mathbb{R}^d$, which is star-shaped with respect to zero, and some bounded function, $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{R})$, with bounded partial derivatives of any order such that the following holds true:*

(i) For all $x \in \mathbb{R}^d$, we have $|\nabla\varphi(x)| < 1$ and

$$H(x, \nabla\varphi(x)) \leqslant 0\,, \quad \text{and} \quad H(x, \nabla\varphi(x)) = 0 \;\Leftrightarrow\; x \in K_0\,. \tag{2.12}$$

(ii) $\varphi(x) - \varphi(y_\star) = d_{\mathrm{A}}(x, y_\star)$, for all x on the geodesic segment from $y_\star$ to $x_\star$.

(iii) $\exp_{y_0}\restriction_{\mathscr{W}} \in C^1(\mathscr{W}, \mathbb{R}^d) \cap C^\infty(\mathscr{W} \setminus \{0\}, \mathbb{R}^d)$ is injective on $\mathscr{W}$ and

$$K_0 \subset \exp_{y_0}(\mathscr{W}) \setminus \{y_0\}\,.$$

(iv) For every $x \in K_0$, there is a unique pair $(\tau, p_0) \in (0,\infty) \times \mathbb{f}_{y_0}$ such that the projection of $[0,\tau] \ni t \mapsto \Phi(t, y_0, p_0)$ onto $\mathbb{R}^d_x$ is a minimizing geodesic from y_0 to x. We have

$$\Phi(\tau, y_0, p_0) \;=\; (x, \nabla\varphi(x)) \;=\; \big(X(\tau, y_0, p_0), \nabla\varphi(X(\tau, y_0, p_0))\big)\,. \tag{2.13}$$

Proof. We proved this proposition in [5] assuming that $H \in C^\infty(\mathbb{R}^{2d}, \mathbb{R})$ and, for all $x \in \mathbb{R}^d$, the function $H(x, \cdot\,) : \mathbb{R}^d \to \mathbb{R}$ is strictly convex, even, and $H(x, 0) < 0$. On account of (1.6) we can easily modify H such that these conditions are satisfied. To this end we first restrict H given by (2.9) to the set $\mathbb{R}^d \times \{p \in \mathbb{R}^d : |p| \leqslant 1 - \delta^2/2\}$ and then we pick an arbitrary smooth extension to $\mathbb{R}^{2d}$ of this restriction fulfilling the condition imposed in [5]. The assertions of the present proposition do not depend on the choice of the latter extension. Another condition required in [5] is that

$$\inf\big\{\, F(x, \mathring{v}) \;:\; x \in \mathbb{R}^d\,,\ \mathring{v} \in S^{d-1}\,\big\} \;>\; 0\,, \tag{2.14}$$

where $F : \mathbb{R}^{2d} \to \mathbb{R}$ is the Finsler structure given by $F(x, v) := \langle\, v \,|\, G(x)\, v\,\rangle^{1/2} = \sqrt{1 - V^2(x)}\,|v|$, $x, v \in \mathbb{R}^d$. Of course, (2.14) is a trivial consequence of (1.6). $\square$

2.3. **Consequences for the symbol of the conjugated Dirac operator.** In this subsection we collect some important properties of the eigenvalues of the symbol

$$\widehat{D}_{V,\varphi}(x,\xi) := \widehat{D}_V(x,\xi+i\nabla\varphi(x))\,, \qquad (x,\xi)\in\mathbb{R}^{2d}, \tag{2.15}$$

always assuming that V fulfills Hypothesis 1.1, $x_\star$ and $y_\star$ fulfill Hypothesis 1.2, and that φ is the function provided by Proposition 2.1, for $d=1$, or Proposition 2.2, for $d\geqslant 2$. We introduce the complex-valued symbols

$$\begin{aligned} a_\pm(x,\xi) &:= \mp i\lambda_\pm\big(x,\xi+i\nabla\varphi(x)\big) \\ &= -i\sqrt{1+(\xi+i\nabla\varphi(x))^2}\mp iV(x)\,, \qquad (x,\xi)\in\mathbb{R}^{2d}. \end{aligned} \tag{2.16}$$

Notice that, since $|\nabla\varphi|<1$, a_+ and a_- are well-defined, complex-valued smooth functions on $\mathbb{R}^{2d}$. In the next two lemmata we collect some basic properties of $a_\pm$ which are used in the sequel. Henceforth, we abbreviate $(a_\pm)''_{x\xi} := d_\xi\nabla_x a_\pm$, $H''_{px} := d_x\nabla_p H$, etc.

Lemma 2.3. *For all $x,\xi\in\mathbb{R}^d$,*

$$\Im a_+(x,\xi) \leqslant 0\,, \tag{2.17}$$

$$\Im a_+(x,\xi) = 0 \quad\Leftrightarrow\quad (x,\xi)\in K_0\times\{0\}\,. \tag{2.18}$$

Moreover, we have, for all $x\in\mathbb{R}^d$,

$$a_+(x,0) = iH(x,\nabla\varphi(x))\,, \tag{2.19}$$

$$\nabla_\xi a_+(x,0) = \nabla_p H(x,\nabla\varphi(x))\,, \tag{2.20}$$

$$(a_+)''_{\xi x}(x,0) = H''_{px}(x,\nabla\varphi(x)) + H''_{pp}(x,\nabla\varphi(x))\,\varphi''(x)\,, \tag{2.21}$$

$$(a_+)''_{\xi\xi}(x,0) = -iH''_{pp}(x,\nabla\varphi(x))\,. \tag{2.22}$$

In particular, for all $x\in K_0$,

$$a_+(x,0)=0\,, \quad \nabla_x a_+(x,0)=0\,, \quad (a_+)''_{xx}(x,0)=0\,, \tag{2.23}$$

$$\nabla_\xi a_+(x,0) = -\nabla\varphi(x)/V(x)\neq 0\,. \tag{2.24}$$

Proof. By virtue of (2.8) and Proposition 2.2(i) we obtain, for $x,\xi\in\mathbb{R}^d$,

$$\begin{aligned} \Im a_+(x,\xi) &= -\Re\sqrt{1-(\nabla\varphi(x))^2+\xi^2+2i\,\langle\,\xi\,|\,\nabla\varphi(x)\,\rangle} - V(x) \\ &\leqslant -\sqrt{1-(\nabla\varphi(x))^2+\xi^2} - V(x) \leqslant 0\,. \end{aligned}$$

Moreover, by (2.12) and the previous inequalities, $\Im a_+(x,\xi)=0$ if and only if $\xi=0$ and $x\in K_0$, which yields (2.17) and (2.18). Furthermore, $\nabla_p H(x,p) = -\nabla_p\lambda_+(x,ip) = -i\nabla_\xi\lambda_+(x,ip)$, for $x\in\mathbb{R}^d$, $|p|<1$, whence $\nabla_p H(x,\nabla\varphi(x)) = \nabla_\xi a_+(x,0)$, for every $x\in\mathbb{R}^d$, which is (2.20). (2.21) follows from (2.20). Next, $H''_{pp}(x,p) = -id_p\nabla_\xi\lambda_+(x,ip) = (\lambda_+)''_{\xi\xi}(x,ip)$, $x\in\mathbb{R}^d$, $|p|<1$, which implies (2.22). All remaining identities are obvious. □

Lemma 2.4. *For all* $x, \xi \in \mathbb{R}^d$,

$$\Im a_-(x,\xi) \leqslant -2\delta < 0\,. \tag{2.25}$$

Proof. Since $H(x, \nabla\varphi) \leqslant 0$ on $\mathbb{R}^d$ the inequality (2.8) implies

$$\begin{aligned}\Im a_-(x,\xi) &= -\Re\sqrt{1-(\nabla\varphi(x))^2+\xi^2+2i\,\langle\,\xi\,|\,\nabla\varphi(x)\,\rangle} + V(x)\\ &\leqslant -\sqrt{1-(\nabla\varphi(x))^2}+V(x) \leqslant 2V(x) \leqslant -2\,\delta\,,\end{aligned}$$

for all $x, \xi \in \mathbb{R}^d$. □

3. The time-dependent complex Hamilton-Jacobi equation

In this section we solve the complex Hamilton-Jacobi equation for $\psi_\pm(t,x,\eta)$,

$$\partial_t\psi_\pm + a_\pm\big(x, \nabla_x\psi_\pm\big) = \mathcal{O}\big((\Im\psi_\pm)^N\big)\,, \quad N \in \mathbb{N}\,, \tag{3.1}$$

$$\psi_\pm(0,x,\eta) = \langle\,\eta\,|\,x\,\rangle\,, \qquad \Im\psi_\pm \geqslant 0\,. \tag{3.2}$$

To this end we proceed along the lines of the constructions in [7]. Instead of solving the problem first for symbols which are homogeneous of degree one as in [7] and then using a standard reduction to that case (see, e.g., [9]) we carry through all constructions for general symbols as in [4]; see also [5] for symbols which are periodic in the momentum variable. Since $a_\pm$ is complex-valued we can only hope to solve (3.1) up error terms $\mathcal{O}((\Im\psi_\pm)^N)$. These do, however, not any harm in the WKB construction as we shall see in the proof of Proposition 4.8; see, in particular, (4.41). In order to control the derivatives of these error terms and of certain implicit functions appearing in the constructions we give some arguments alternative to those in [7]. The final result of this section is Corollary 3.10 where (3.1)&(3.2) is solved.

In the whole Section 3 we always assume that V fulfills Hypothesis 1.1 and $x_\star$, $y_\star$ fulfill Hypothesis 1.2. φ is the function provided by Proposition 2.1, for $d = 1$, or Proposition 2.2, for $d \geqslant 2$.

3.1. **Estimates on the Hamiltonian and contact flows.** From now on the symbols x, y, ξ, and η will denote complex variables in $\mathbb{C}^d$. We set $\rho = (x,\xi)$ and $\partial_{x_j} = (\partial_{\Re x_j} - i\partial_{\Im x_j})/2$, $\partial_{\overline{x}_j} = (\partial_{\Re x_j} + i\partial_{\Im x_j})/2$, $\nabla_x = (\nabla_{\Re x} - i\nabla_{\Im x})/2$, $\nabla_{\overline{x}} = (\nabla_{\Re x} + i\nabla_{\Im x})/2$, ..., and analogously for complex variables other than x. In the rest of this article we further extend φ and V almost analytically to smooth functions defined on $\mathbb{C}^d$ – again denoted by the symbols φ and V – so that, for every compact subset $K \subset \mathbb{C}^d$ and all $N \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^{2d}$, we find some $C_{N,K,\alpha} \in (0,\infty)$ such that

$$\begin{aligned}|\partial^\alpha_{(\Re x, \Im x)}\nabla_{\overline{x}}\varphi(x)| &\leqslant C_{N,K,\alpha}\,|\Im x|^N, \qquad x \in K\,,\\ |\partial^\alpha_{(\Re x, \Im x)}\nabla_{\overline{x}}V(x)| &\leqslant C_{N,K,\alpha}\,|\Im x|^N, \qquad x \in K\,.\end{aligned}$$

Then we find some open neighborhood, $\Omega \subset \mathbb{C}^{2d}$, of $\mathbb{R}^{2d}$ such that the symbols

$$a_\pm(x,\xi) = -i\sqrt{1+(\xi+i\nabla\varphi(x))^2} \mp iV(x)\,, \qquad (x,\xi)\in\Omega\,,$$

are well-defined and almost analytic on Ω. For every compact subset $K \subset \Omega$ and all $N \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^{4d}$, we find some $C_{N,K,\alpha} \in (0,\infty)$ such that

$$|\partial^\alpha_{(\Re\rho,\Im\rho)}\nabla_{\overline{\rho}} a_\pm(\rho)| \leqslant C_{N,K,\alpha}\,|\Im\rho|^N, \quad \rho\in K\,. \tag{3.3}$$

(In fact, $\nabla_{\overline{\xi}} a_\pm = 0$ on Ω.) In this subsection we proceed along the lines of [4, 7] to obtain estimates on the Hamiltonian and contact flows associated with $a_\pm$. On Ω we introduce the Hamiltonian vector fields

$$\mathscr{H}_{a_\pm} := \langle\, \nabla_\xi a_\pm \,|\, \nabla_x \,\rangle - \langle\, \nabla_x a_\pm \,|\, \nabla_\xi \,\rangle\,,$$

and the elementary actions

$$\mathscr{A}_\pm(\rho) := \langle\, \nabla_\xi a_\pm(\rho)\,|\,\xi\,\rangle - a_\pm(\rho)\,, \qquad \rho=(x,\xi)\in\Omega\,.$$

Here and henceforth $\langle\,\cdot\,|\,\cdot\cdot\,\rangle$ denotes the extension of the Euclidean scalar product to a *bilinear* form on $\mathbb{C}^d$. For later reference we infer from (2.23) that

$$\mathscr{A}_+(x,0) = 0\,, \qquad x\in K_0\,. \tag{3.4}$$

We further add an extra variable, $s\in\mathbb{C}$, to $(x,\xi)\in\Omega$ which parameterizes the action and define the contact fields

$$\mathscr{K}_{a_\pm} := -\mathscr{A}_\pm\,\partial_s + \mathscr{H}_{a_\pm} = (a_\pm - \langle\, \nabla_\xi a_\pm \,|\, \xi \,\rangle)\,\partial_s + \mathscr{H}_{a_\pm} \qquad \text{on } \mathbb{C}\times\Omega\,.$$

Finally, we introduce the real partial differential operators

$$\widehat{\mathscr{H}_{a_\pm}} := \mathscr{H}_{a_\pm} + \overline{\mathscr{H}}_{a_\pm}\,, \qquad \widehat{\mathscr{K}_{a_\pm}} := \mathscr{K}_{a_\pm} + \overline{\mathscr{K}}_{a_\pm}\,.$$

Notice that, since $c\,\partial_z + \overline{c}\,\partial_{\overline{z}} = (\Re c)\,\partial_{\Re z} + (\Im c)\,\partial_{\Im z}$, the vector in $\mathbb{C}^{2d}$ corresponding to $\widehat{\mathscr{H}_{a_\pm}}$ under the identification $\partial_{\Re x_j} \leftrightarrow \mathsf{e}_j$, $\partial_{\Im x_j} \leftrightarrow i\mathsf{e}_j$, $\partial_{\Re \xi_j} \leftrightarrow \mathsf{e}_{d+j}$, $\partial_{\Im \xi_j} \leftrightarrow i\mathsf{e}_{d+j}$, $j=1,\dots,d$, where $(\mathsf{e}_1,\dots,\mathsf{e}_{2d})$ is the canonical basis of $\mathbb{C}^{2d}$, is just

$$\widehat{\mathscr{H}_{a_\pm}} \leftrightarrow \begin{pmatrix} \nabla_\xi a_\pm \\ -\nabla_x a_\pm \end{pmatrix}. \tag{3.5}$$

We denote the flow of $\widehat{\mathscr{H}_{a_\pm}}$ as $\kappa_t^\pm = (Q^\pm,\Xi^\pm) : \mathcal{D}(\kappa^\pm) \to \mathbb{C}^{2d}$, so that $\mathcal{D}(\kappa^\pm) = \{(t,y,\eta)\in\mathbb{R}\times\Omega : t\in J^\pm_{\max}(y,\eta)\}$, where $J^\pm_{\max}(y,\eta)$ denotes the maximal interval of existence for the $4d$-dimensional real initial value problem $\dot\rho = \widehat{\mathscr{H}_{a_\pm}}(\rho)$, $\rho(0)=(y,\eta)$, and

$$\partial_t\kappa^\pm = \partial_t\begin{pmatrix} Q^\pm \\ \Xi^\pm \end{pmatrix} = \begin{pmatrix} \nabla_\xi a_\pm(Q^\pm,\Xi^\pm) \\ -\nabla_x a_\pm(Q^\pm,\Xi^\pm) \end{pmatrix} \tag{3.6}$$

on $\mathcal{D}(\kappa^\pm)$. The flow of $\widehat{\mathscr{K}_{a_\pm}}$ is then given by $(\varsigma^\pm, \kappa^\pm) : \mathbb{C} \times \mathcal{D}(\kappa^\pm) \to \mathbb{C}^{1+2d}$, where

$$\varsigma^\pm(s,t,y,\eta) := \varsigma_t^\pm(s,y,\eta) := s - \int_0^t \mathscr{A}_\pm(\kappa_r^\pm(y,\eta))\,dr, \quad s \in \mathbb{C}\,,\ (t,y,\eta) \in \mathcal{D}(\kappa^\pm).$$

Lemma 3.1. *Let $x_0 \in K_0$ and $I \subset I_{\max}(x_0, \nabla\varphi(x_0))$ some interval such that $X(t, x_0, \nabla\varphi(x_0)) \in K_0$, for all $t \in I$. Then $I \subset J^+_{\max}(x_0, 0)$ and*

$$Q^+(t, x_0, 0) \,=\, X\big(t, x_0, \nabla\varphi(x_0)\big)\,, \quad \Xi^+(t, x_0, 0) \,=\, 0\,, \qquad t \in I\,.$$

(Recall the notation introduced above (2.11)*.)*

Proof. By (2.20) and (2.23) we have $\widehat{\mathscr{H}_{a_+}} = \nabla_p H(x, \nabla\varphi) \cdot \nabla_{\Re x}$ on $K_0 \times \{0\}$. Moreover, $\partial_t X_t(x_0, \nabla\varphi(x_0)) = \nabla_p H(\Phi_t(x_0, \nabla\varphi(x_0)))$, and $\Phi_t(x_0, \nabla\varphi(x_0)) = (X_t, \nabla\varphi(X_t))(x_0, \nabla\varphi(x_0))$, $t \in I$, by (2.13). ☐

Lemma 3.2. *Let $\tau > 0$ and assume that $\rho : [0, \tau] \to \mathbb{R}^{2d}$ is a real integral curve of $\widehat{\mathscr{H}_{a_+}}$ with $\rho(0) \in K_0 \times \{0\} = \{\Im a_+ = 0\}$. Then $\rho([0, \tau]) \subset K_0 \times \{0\}$.*

Proof. By assumption $\Im(\nabla_\xi a_+, -\nabla_x a_+)(\rho(t)) = \frac{d}{dt}\Im\rho(t) = 0$. Since a_+ fulfills the Cauchy-Riemann differential equations on the real domain, it follows that $(\Im a_+)'_{\Re\rho}(\rho(t)) = \Im(a_+)'_\rho(\rho(t)) = 0$. Hence, the derivative of $(\Im a_+)\restriction_{\mathbb{R}^{2d}}$ vanishes along ρ, thus $\Im a_+(\rho(0)) = 0$ implies $\rho([0, \tau]) \subset \{\Im a_+ = 0\}$. Using (2.18) we conclude $\rho([0, \tau]) \subset K_0 \times \{0\}$. ☐

In what follows we consider the trajectories of $\widehat{\mathscr{K}_{a_\pm}}$ emanating from the planes

$$\mathfrak{L}_0(\eta) \,:=\, \big\{(-\psi_0(y,\eta), y, \nabla_y\psi_0(y,\eta)) \,:\, y \in \mathbb{C}^d\,\big\}\,,$$

where $\psi_0(y,\eta) = \langle\,\eta\,|\,y\,\rangle$, $y \in \mathbb{C}^d$, $\eta \in \mathbb{R}^d$, so that $\nabla_y\psi_0(y,\eta) = \eta$. The reason why we restrict our attention to real η is that in this case ψ_0 trivially fulfills the inequality

$$\Im\psi_0(y,\eta) - \langle\,\Im y\,|\,\Re\nabla_y\psi_0(y,\eta)\,\rangle \,\geqslant\, -\mathcal{O}\big(|\Im(y,\eta)|^3\big)\,, \quad y \in \mathbb{C}^d, \tag{3.7}$$

which is used to derive the estimates of Lemma 3.3 below. (We could equally well consider more general ψ_0 satisfying (3.7) locally on compact subsets in Lemma 3.3.) We define $\mathfrak{S} : \mathbb{C}^{1+2d} \to \mathbb{R}$ by

$$\mathfrak{S}(s,x,\xi) := -\Im s - \langle\,\Im x\,|\,\Re\xi\,\rangle = -\Im\big(s + \langle\,x\,|\,\Re\xi\,\rangle\big), \quad (s,x,\xi) \in \mathbb{C}^{1+2d}, \tag{3.8}$$

which corresponds to the function $-\langle\,\Im x\,|\,\Re\xi\,\rangle$ considered in [7] where the symbol is homogeneous of degree one in ξ. The following lemma is a slight modification of [7, Proposition 3.1]. For the convenience of the reader we present its proof in Appendix B.

Lemma 3.3. *Let $y_0, \eta_0 \in \mathbb{R}^d$. In the minus-case we set $\tau = 0$. In the plus-case we pick some $\tau \in J^+_{\max}(y_0, \eta_0)$, $\tau \geqslant 0$. If $\tau > 0$ we assume that $\kappa^+_t(y_0, \eta_0)$ is* real, *for all $t \in [0, \tau]$, and $\Im a_+(y_0, \eta_0) = 0$. Then there exist $\varepsilon > 0$, $C \in (0, \infty)$, and some neighborhood $\mathscr{O} \subset \mathbb{C}^{1+2d}$ of $(-\psi_0(y_0, \eta_0), y_0, \eta_0)$ such that the following inequalities are satisfied on $\mathfrak{L}_0(\eta_0) \cap \mathscr{O}$, for all $0 \leqslant r \leqslant t \leqslant \tau + \varepsilon$ and $h \in [0, 1]$,*

$$\mathfrak{S}(\varsigma^\pm_t, \kappa^\pm_t) \geqslant \frac{1}{2} \int_0^t -\Im a_\pm(\Re \kappa^\pm_u)\, du - C\, |\Im \kappa^\pm_t|^3, \tag{3.9}$$

$$|\Im \kappa^\pm_t|^2 + \mathfrak{S}(\varsigma^\pm_t, \kappa^\pm_t) \geqslant \frac{1}{C} \Big\{ |\Im \kappa^\pm_r|^2 + \mathfrak{S}(\varsigma^\pm_r, \kappa^\pm_r) + \int_r^t -\Im a_\pm(\Re \kappa^\pm_u)\, du \Big\}, \tag{3.10}$$

$$|\Im \kappa^\pm_r|^2 \leqslant C \left(|\Im \kappa^\pm_t|^2 + \mathfrak{S}(\varsigma^\pm_t, \kappa^\pm_t) \right). \tag{3.11}$$

The constants ε and C can be chosen uniform when η_0 varies in some compact set.

In particular, if $(s, \rho) \in \mathfrak{L}_0(\eta_0) \cap \mathscr{O}$ and $\kappa^\pm_t(\rho)$ and $\varsigma^\pm_t(s, \rho)$ are both real, for some $t \in [0, \tau + \varepsilon]$, then $\kappa^\pm_r(\rho)$ is real for all $r \in [0, t]$.

We recall that the Hamilton matrix, $\mathbb{F}_{a_\pm}$, of $a_\pm$ is given as (recall (3.5))

$$\mathbb{F}_a\, \theta = \begin{pmatrix} a''_{\xi x} & a''_{\xi\xi} \\ -a''_{xx} & -a''_{x\xi} \end{pmatrix} \begin{pmatrix} \theta_x \\ \theta_\xi \end{pmatrix} + \begin{pmatrix} a''_{\xi\overline{x}} & a''_{\xi\overline{\xi}} \\ -a''_{x\overline{x}} & -a''_{x\overline{\xi}} \end{pmatrix} \begin{pmatrix} \overline{\theta}_x \\ \overline{\theta}_\xi \end{pmatrix}, \quad a \in \{a_+, a_-\}, \tag{3.12}$$

where $\theta = (\theta_x, \theta_\xi) \in \mathbb{C}^{2d}$ and where the matrix on the right vanishes on the real domain $\mathbb{R}^{2d} \subset \Omega$. Here and henceforth we write $a''_{x\overline{\xi}} = \frac{1}{4}(d\Re\xi + i d\Im\xi)(\nabla_{\Re x} - i\nabla_{\Im x})\, a$, etc. Let $\mathscr{I} : \mathbb{C}^{2d} \to \mathbb{C}^{2d}$ denote multiplication with i. Then (3.3) implies that, for every compact $K \subset \Omega$ and all $N \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^{4d}$, there is some $C_{N,K,\alpha} \in (0, \infty)$ such that

$$\frac{1}{2} \left\| \partial^\alpha_{(\Re\rho, \Im\rho)} [\mathscr{I}, \mathbb{F}_a] \right\| = \left\| \partial^\alpha_{(\Re\rho, \Im\rho)} \begin{pmatrix} a''_{\xi\overline{x}} & a''_{\xi\overline{\xi}} \\ -a''_{x\overline{x}} & -a''_{x\overline{\xi}} \end{pmatrix} \right\| \leqslant C_{N,K,\alpha}\, |\Im\rho|^N \tag{3.13}$$

on K, where a again is a_+ or a_-.

Corollary 3.4. *Let (y_0, η_0), τ, and ε be as in Lemma 3.3, set $T := \tau + \varepsilon$, and let $K \subset \mathbb{C}^{2d}$ be some sufficiently small compact neighborhood of (y_0, η_0). Then, for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^{4d+1}$, $\beta \in \mathbb{N}_0^{3d+1}$, there exist $C_{N,K,T,\alpha}, C'_{N,K,T,\beta} \in (0, \infty)$ such that*

$$\left\| \partial^\alpha_{(t, \Re\rho, \Im\rho)} d_{\overline{\rho}} \kappa^\pm_t(\rho) \right\| \leqslant C_{N,K,T,\alpha} \sup_{s \in [0,t]} |\Im \kappa^\pm_s(\rho)|^N, \quad \rho \in K,\ t \in [0, T], \tag{3.14}$$

and, for $(y, \eta) \in K \cap (\mathbb{C}^d \times \mathbb{R}^d)$, and $t \in [0, T]$,

$$\left\| \partial^\beta_{(t, \Re y, \Im y, \Re\eta)} d_{(\overline{y}, \overline{\eta})} \kappa^\pm_t(y, \eta) \right\| \leqslant C'_{N,K,T,\beta} \big(|\Im \kappa^\pm_t(y, \eta)|^2 + \mathfrak{S}(\varsigma^\pm_t, \kappa^\pm_t)(y, \eta) \big)^N. \tag{3.15}$$

Proof. Again we drop all $\pm$-indices in this proof. If $\mathscr{I}$ denotes multiplication by i, we have

$$\big\|\partial^\alpha_{(t,\Re\rho,\Im\rho)} d_{\overline{\rho}}\kappa_t(\rho)\big\| = \frac{1}{2}\,\big\|\,[\mathscr{I}\,,\,\partial^\alpha_{(t,\Re\rho,\Im\rho)}\kappa'_t(\rho)\,]\big\|\,,\quad \rho\in K.$$

Moreover, we know that $\kappa'_t(\rho)$, $\rho\in K$, satisfies $\frac{d}{dt}\,\kappa'_t(\rho) = \mathbb{F}_a(\kappa_t(\rho))\,\kappa'_t(\rho)$, $t\in[0,T]$, $\kappa'_s(\kappa'_t(\rho)) = \kappa'_{t+s}(\rho)$, $t, t+s\in[0,T]$, and $\kappa'_0(\rho) = \mathbb{1}$. In particular, $[\mathscr{I}\,,\,\kappa'_0(\rho)] = 0$. Since we have

$$\begin{aligned}\frac{d}{dt}\,[\mathscr{I}\,,\kappa'_t(\rho)] \;&=\; \big[\mathscr{I}\,,\,\mathbb{F}_a(\kappa_t(\rho))\,\kappa'_t(\rho)\big]\\ &=\; \mathbb{F}_a(\kappa_t(\rho))\,[\mathscr{I}\,,\kappa'_t(\rho)] \,+\, [\mathscr{I}\,,\mathbb{F}_a(\kappa_t(\rho))]\,\kappa'_t(\rho)\,,\qquad t\in[0,T],\end{aligned}$$

it thus follows from Duhamel's formula that

$$[\mathscr{I}\,,\kappa'_t(\rho)] \;=\; \int_0^t \kappa'_{t-s}(\rho)\,[\mathscr{I}\,,\mathbb{F}_a(\kappa_s(\rho))]\,\kappa'_s(\rho)\,ds\,,\qquad t\in[0,T],\ \rho\in K.$$

Using $\sup_{t\in[0,T]}\sup_{\rho\in K}\|\kappa'_t(\rho)\|<\infty$, and $[\mathscr{I}\,,\mathbb{F}_a(\kappa_s(\rho))] = \mathcal{O}\big(|\Im\kappa_s(\rho)|^N\big)$, $N\in\mathbb{N}$, we deduce that the following estimate is satisfied in the case $\alpha=0$,

$$\big\|[\mathscr{I}\,,\partial^\alpha_{(\Re\rho,\Im\rho)}\kappa'_t(\rho)]\big\| \leqslant C'_{K,N,T,\alpha}\sup_{s\in[0,T]}|\Im\kappa_s(\rho)|^N,\qquad \rho\in K\,,\tag{3.16}$$

for some $C'_{K,N,T,\alpha}\in(0,\infty)$. If (3.16) holds true, for all multi-indices of length $\leqslant n\in\mathbb{N}_0$, and $|\alpha| = n+1$, we write

$$\begin{aligned}\frac{d}{dt}\,[\mathscr{I}\,,\partial^\alpha_{(\Re\rho,\Im\rho)}\kappa'_t] &= \mathbb{F}_a(\kappa_t)\,[\mathscr{I}\,,\partial^\alpha_{(\Re\rho,\Im\rho)}\kappa'_t] + [\mathscr{I}\,,\mathbb{F}_a(\kappa_t)]\,\partial^\alpha_{(\Re\rho,\Im\rho)}\kappa'_t\\ +\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}&\big\{[\mathscr{I}\,,\partial^\beta_{(\Re\rho,\Im\rho)}\mathbb{F}_a(\kappa_t)]\,\partial^{\alpha-\beta}_{(\Re\rho,\Im\rho)}\kappa'_t + \partial^\beta_{(\Re\rho,\Im\rho)}\mathbb{F}_a(\kappa_t)\,[\mathscr{I}\,,\partial^{\alpha-\beta}_{(\Re\rho,\Im\rho)}\kappa'_t]\big\}.\end{aligned}\tag{3.17}$$

Here we know that $\sup_{t\in[0,T]}\sup_{\rho\in K}\|\partial^{\alpha-\beta}_{(\Re\rho,\Im\rho)}\kappa'_t(\rho)\|<\infty$, for $0\leqslant\beta\leqslant\alpha$, and $\|\,[\mathscr{I}\,,\partial^\beta_{(\Re\rho,\Im\rho)}\mathbb{F}_a(\kappa_t)]\,\| = \mathcal{O}\big(|\Im\kappa_t(\rho)|^N\big)$ by (3.13). By the induction hypothesis $\|\,[\mathscr{I}\,,\partial^{\alpha-\beta}_{(\Re\rho,\Im\rho)}\kappa'_t]\,\| = \mathcal{O}\big(\sup_{s\in[0,T]}|\Im\kappa_s(\rho)|^N\big)$, for $0<\beta\leqslant\alpha$. Applying Duhamel's formula once more, using $[\mathscr{I}\,,\partial^\alpha_{(\Re\rho,\Im\rho)}\kappa'_0] = 0$, we obtain (3.16), for $\alpha\in\mathbb{N}_0^{4d}$, $|\alpha| = n+1$. Taking successively time derivatives of (3.17) we can apply a bootstrap argument to include higher order time derivatives and to get

$$\big\|[\mathscr{I}\,,\partial^\alpha_{(t,\Re\rho,\Im\rho)}\kappa'_t(\rho)]\big\| \leqslant C''_{K,N,T,\alpha}\sup_{s\in[0,T]}|\Im\kappa_s(\rho)|^N,\qquad \rho\in K\,,\tag{3.18}$$

for some $C''_{K,N,T,\alpha}\in(0,\infty)$ and all $\alpha\in\mathbb{N}_0^{4d+1}$. Combining (3.18) with (3.11) we arrive at the asserted estimates (3.14) and (3.15). $\square$

3.2. **Approximate solution of the complex Hamilton-Jacobi equation.** We recall the notation (3.6) and set

$$\begin{aligned}
\widetilde{\mathscr{D}}^+ &:= \left\{ (t,y,0) \in \mathbb{R}^+_0 \times K_0 \times \{0\} \ : \ Q^+_{t'}(y,0) \in K_0 \,,\ t' \in [0,t] \right\}, \\
\mathscr{D}^+ &:= \left\{ (t,x,0) \in \mathbb{R}^+_0 \times K_0 \times \{0\} \ : \ Q^+_{-t'}(x,0) \in K_0 \,,\ t' \in [0,t] \right\}, \\
\widetilde{\mathscr{E}}^+ &:= (\{0\} \times \mathbb{R}^{2d}) \cup \widetilde{\mathscr{D}}^+, \qquad \mathscr{E}^+ := (\{0\} \times \mathbb{R}^{2d}) \cup \mathscr{D}^+, \\
\mathscr{F}^+ &:= \left\{ (t, Q^+_t(y,0), 0, y, 0) \ : \ (t,y,0) \in \widetilde{\mathscr{D}}^+ \right\} \cup \left\{ (0,x,\eta,x,\eta) \ : \ x,\eta \in \mathbb{R}^d \right\}.
\end{aligned}$$

We can represent $\mathscr{F}^+$ as the graph of the function $(k^+, g^+) : \mathscr{E}^+ \to \mathbb{R}^{2d}$, where

$$k^+(t,x,0) = Q^+(-t,x,0) = X(-t,x,\nabla\varphi(x)), \quad (t,x,0) \in \mathscr{D}^+, \tag{3.19}$$

$$k^+(0,x,\eta) = x, \qquad (x,\eta) \in \mathbb{R}^{2d}, \tag{3.20}$$

and

$$g^+(t,x,0) = 0\,, \ \ (t,x,0) \in \mathscr{D}^+, \qquad g^+(0,x,\eta) = \eta\,, \ \ (x,\eta) \in \mathbb{R}^{2d}. \tag{3.21}$$

In the minus-case we simply set

$$\widetilde{\mathscr{E}}^- := \mathscr{E}^- := \{0\} \times \mathbb{R}^{2d}, \quad \mathscr{F}^- := \left\{ (0,x,\eta,x,\eta) \ : \ x,\eta \in \mathbb{R}^d \right\}.$$

Then $\mathscr{F}^-$ is the graph of $(k^-, g^-) : \mathscr{E}^- \to \mathbb{R}^{2d}$, where

$$k^-(0,x,\eta) = x\,, \quad g^-(0,x,\eta) = \eta\,, \qquad (x,\eta) \in \mathbb{R}^{2d}. \tag{3.22}$$

The next lemma shows that we can extend $k^\pm$ and $g^\pm$ to smooth functions defined in a neighborhood of $\mathscr{E}^\pm$ which represent the canonical relations given by the flow of $\widehat{\mathscr{H}_{a_\pm}}$ as a graph in the vicinity of $\mathscr{F}^\pm$.

Lemma 3.5. *There exist open neighborhoods, $\mathscr{G}^\pm$ of $\overline{\mathscr{E}}^\pm$ in $\mathbb{R}^+_0 \times \mathbb{C}^{2d}$ and $\mathscr{H}^\pm$ of $\overline{\mathscr{F}}^\pm$ in $\mathbb{R}^+_0 \times \mathbb{C}^{2d} \times \Omega$, and $k^\pm, g^\pm \in C^\infty(\mathscr{G}^\pm, \mathbb{C}^d)$, such that, for all $(t,x,\xi,y,\eta) \in \mathscr{H}^\pm$,*

$$(x,\xi) = \left(Q^\pm_t(y,\eta), \Xi^\pm_t(y,\eta) \right) \quad \Leftrightarrow \quad \left(y = k^\pm(t,x,\eta) \ \wedge \ \xi = g^\pm(t,x,\eta) \right).$$

Proof. We define

$$F^\pm(t,x,\xi,y,\eta) := \left(x - Q^\pm(t,y,\eta)\,,\, \xi - \Xi^\pm(t,y,\eta) \right), \tag{3.23}$$

for all $(x,\xi,t,y,\eta) \in \mathbb{C}^{2d} \times \mathcal{D}(\kappa^\pm)$. In the following we regard $F^\pm$ as a $\mathbb{R}^{4d}$-valued function of $8d+1$ real variables. At $t=0$ we have $F^\pm(0,x,\xi,y,\eta) = (x-y, \xi-\eta)$ and it is trivial that $(F^\pm)'_{(\Re\xi,\Im\xi,\Re y,\Im y)}|_{t=0} : \mathbb{R}^{4d} \to \mathbb{R}^{4d}$ is invertible. This already proves the assertion in the minus-case. In the plus-case we have, for general $(x,\xi,t,y,\eta) \in \mathbb{C}^{2d} \times \mathcal{D}(\kappa^+)$,

$$(F^+)'_{(\Re\xi,\Im\xi,\Re y,\Im y)}(t,x,\xi,y,\eta) \ = \ \begin{pmatrix} 0_{2d} & -(Q^+)'_{(\Re y,\Im y)} \\ \mathbb{1}_{2d} & -(\Xi^+)'_{(\Re y,\Im y)} \end{pmatrix} (t,y,\eta)\,.$$

We know, however, that $Q^+(t,y,0) = X(t,y,\nabla\varphi(y))$, for $(t,y,0) \in \widetilde{\mathscr{D}}^+$, where we use the notation introduced in the paragraph preceding (2.11). In view of (3.16) we further have $(\Im Q^+)'_{\Im y}(t,y,0) = (\Re Q^+)'_{\Re y}(t,y,0) = d_y[X(t,y,\nabla\varphi(y))]$ and $(\Re Q^+)'_{\Im y}(t,y,0) = -(\Im Q^+)'_{\Re y}(t,y,0) = 0$, for $(t,y,0) \in \widetilde{\mathscr{D}}^+$. Likewise we have $\Xi^+(t,y,0) = 0$ and, hence, $(\Xi^+)'_{(\Re y,\Im y)}(t,y,0) = 0$, for $(t,y,0) \in \widetilde{\mathscr{D}}^+$. It follows that

$$(F^+)'_{(\Re\xi,\Im\xi,\Re y,\Im y)}(t,Q^+(t,y,0),0,y,0) = \begin{pmatrix} 0_{2d} & -d_y[X(t,y,\nabla\varphi(y))] \otimes \mathbb{1}_2 \\ \mathbb{1}_{2d} & 0_{2d} \end{pmatrix},$$

for all $(t,y,0) \in \widetilde{\mathscr{D}}^+$. Moreover, the matrix $d_y[X(t,y,\nabla\varphi(y))]$ is invertible as a time zero fundamental matrix of some matrix-valued ODE. In fact, it holds $X(0,y,\nabla\varphi(y)) = y$, thus $d_y[X(0,y,\nabla\varphi(y))] = \mathbb{1}$, and

$$\partial_t d_y[X(t,y,\nabla\varphi(y))] \,=\, \mathbb{B}(t,y)\, d_y[X(t,y,\nabla\varphi(y))]\,,$$

for every $(t,y,0) \in \widetilde{\mathscr{D}}^+$, where

$$\mathbb{B}(t,y) := \big(H''_{px}\big(x,\nabla\varphi(x)\big) + H''_{pp}\big(x,\nabla\varphi(x)\big)\,\varphi''(x)\big)\big|_{x=X(t,y,\nabla\varphi(y))}\,. \tag{3.24}$$

Since $\mathscr{F}^+$ can globally be represented as a graph we conclude that functions k^+ and g^+ with the properties stated in the assertion exist and are unique if the neighborhood $\mathscr{G}^+$ is chosen sufficiently small. $\square$

In the remaining part of this subsection we show that a solution of (3.1) is given by the following formula well-known from classical mechanics,

$$\begin{aligned} \psi_\pm(t,x,\eta) &:= (Z_\pm \circ K_\pm)(t,x,\eta) \\ &= \langle\, k^\pm(t,x,\eta) \,|\, \eta \,\rangle + \int_0^t \mathscr{A}_\pm\big(\kappa_r^\pm(k^\pm(t,x,\eta),\eta)\big)\,dr\,, \end{aligned} \tag{3.25}$$

for $(t,x,\eta) \in \mathscr{G}^\pm$, where

$$Z_\pm(t,y,\eta) \,:=\, -\varsigma_t^\pm(-\langle\, y \,|\, \eta \,\rangle, y, \eta) \,=\, \langle\, y \,|\, \eta \,\rangle + \int_0^t \mathscr{A}_\pm(\kappa_r^\pm(y,\eta))\,dr\,,$$

for $(t,y,\eta) \in \mathcal{D}(\kappa^\pm)$, and

$$K_\pm(t,x,\eta) \,:=\, \big(t, k^\pm(t,x,\eta), \eta\big)\,, \qquad (t,x,\eta) \in \mathscr{G}^\pm. \tag{3.26}$$

To this end we further introduce the following canonical weights which are used to control the error terms,

$$\begin{aligned} \widetilde{\Gamma}_\pm \,&:=\, |\Im\kappa^\pm|^2 + \mathfrak{S}(-Z_\pm,\kappa^\pm)\,, \quad \text{on } \mathcal{D}(\kappa^\pm)\,, \\ \Gamma_\pm \,&:=\, \widetilde{\Gamma}_\pm \circ K_\pm \,=\, |\Im(x,g^\pm)|^2 - \langle\, \Im x \,|\, \Re g^\pm \,\rangle + \Im\psi_\pm\,, \quad \text{on } \mathscr{G}^\pm. \end{aligned}$$

Here we used that $\kappa_t^\pm(k^\pm(t,x,\eta),\eta) = (x, g^\pm(t,x,\eta))$, for all $(t,x,\eta) \in \mathscr{G}^\pm$.

Lemma 3.6. *There is an open neighborhood,* $\widetilde{\mathscr{N}_\pm} \subset \mathbb{R}_0^+ \times \mathbb{C}^d \times \mathbb{R}^d$*, of the closure of* $\widetilde{\mathscr{E}}^\pm$ *such that, for every compact subset* $K \subset \widetilde{\mathscr{N}_\pm}$*, we find some* $C_K \in (0,\infty)$ *such that, for all* $(t,y,\eta) \in K$ *and* $r \in [0,t]$*,*

$$\begin{aligned} \Im Z_\pm(t,y,\eta) \geqslant \langle\, \Im Q^\pm \,|\, \Re \Xi^\pm \,\rangle(t,y,\eta) - \frac{1}{2}\int_0^t \Im a_\pm(\Re \kappa_s(y,\eta))\, ds \\ - C_K \left|\Im(Q^\pm,\Xi^\pm)(t,y,\eta)\right|^3, \end{aligned} \tag{3.27}$$

$$\frac{1}{2}\left|\Im(Q^\pm,\Xi^\pm)(r,y,\eta)\right|^2 \leqslant \widetilde{\Gamma}_\pm(r,y,\eta) \leqslant C_K\, \widetilde{\Gamma}_\pm(t,y,\eta)\,, \tag{3.28}$$

Proof. We apply Lemma 3.3, for every fixed $\eta_0 \in \mathbb{R}^d$, recalling that the constant C appearing there can be chosen uniform when η_0 varies in a compact set. In the minus case we always choose $\tau = 0$ in Lemma 3.3. In the plus case we choose $\tau = 0$, if $(y_0,\eta_0) \notin K_0 \times \{0\}$. If, however, $\eta_0 = 0$ and $y_0 \in K_0$, then we choose $\tau = \max\{t \geqslant 0 :\ X(r,y_0,\nabla\varphi(y_0)) \in K_0,\ r \in [0,t]\}$. Then all assumptions of Lemma 3.3 are satisfied because $\Im a_+(y_0,0) = 0$ and $\Im\kappa_t(y_0,0) = \Im(X(t,y_0,\nabla\varphi(y_0)),0) = 0$, $t \in [0,\tau]$, by (2.18) and Lemma 3.1, respectively. □

First, we derive some estimates on the derivatives of the implicit functions $k^\pm$ and $g^\pm$. To this end we put

$$\mathscr{N}_\pm := K_\pm^{-1}(\widetilde{\mathscr{N}_\pm})\,, \tag{3.29}$$

where $K_\pm$ is given by (3.26) and $\widetilde{\mathscr{N}_\pm}$ by Lemma 3.6, so that $\mathscr{N}_\pm \subset \mathscr{G}^\pm$ is a neighborhood of $\overline{\mathscr{E}}^\pm$ in $\mathbb{R}_0^+ \times \mathbb{C}^d \times \mathbb{R}_d$.

Lemma 3.7. *Let* $k^\pm, g^\pm \in C^\infty(\mathscr{G}^\pm,\mathbb{C}^d)$ *be the implicit functions provided by Lemma 3.5. Then, for all compact subsets* $K \subset \mathscr{N}^\pm$ *and all* $N \in \mathbb{N}$ *and* $\alpha \in \mathbb{N}_0^{3d+1}$*, there is some* $C_{N,K,\alpha} \in (0,\infty)$ *such that, for all* $(t,x,\eta) \in K$*,*

$$\left\|\partial^\alpha_{(t,\Re x,\Im x,\Re\eta)} d_{\overline{x}}(k^\pm,g^\pm)(t,x,\eta)\right\| \leqslant C_{N,K,\alpha}\, \Gamma_\pm(t,x,\eta)^N.$$

Proof. Dropping all $\pm$-indices and using the notation (3.23) we have

$$\begin{pmatrix} k'_{(\Re x,\Im x)} \\ g'_{(\Re x,\Im x)} \end{pmatrix} = \begin{pmatrix} 0_{2d} & -Q'_{(\Re y,\Im y)} \\ \mathbb{1}_{2d} & -\Xi'_{(\Re y,\Im y)} \end{pmatrix}^{-1} \begin{pmatrix} -\mathbb{1}_{2d} \\ 0_{2d} \end{pmatrix},$$

where all derivatives of Q and Ξ are evaluated at $(t,k(t,x,\eta),\eta)$, for $(t,x,\eta) \in \mathscr{G}$. We denote the above matrices as A, B, and C, so that $A = B^{-1}C$. Let $\mathscr{I}_n$ represent multiplication with i on $\mathbb{C}^d = \mathbb{R}^{2d}$, that is, $\mathscr{I}_n = \mathbb{1}_n \otimes \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Writing $[\mathscr{I},A] := \mathscr{I}_{2d}A - A\mathscr{I}_d$, etc., we then have

$$[\mathscr{I},A] = [\mathscr{I},B^{-1}]\,C + B^{-1}\,[\mathscr{I},C] = B^{-1}\,[B,\mathscr{I}]\,B^{-1}\,C\,. \tag{3.30}$$

Taking derivatives of (3.30) we obtain, for $\alpha \in \mathbb{N}_0^{1+3d}$,

$$[\mathscr{I}, \partial^\alpha_{(t,\Re x,\Im x,\eta)} A]$$
$$= \sum_{\beta+\gamma+\delta=\alpha} c(\beta,\gamma,\delta)\,\{\partial^\beta_{(t,\Re x,\Im x,\eta)} B^{-1}\}[\partial^\gamma_{(t,\Re x,\Im x,\eta)} B, \mathscr{I}]\{\partial^\delta_{(t,\Re x,\Im x,\eta)}(B^{-1}C)\}\,,$$

for some combinatorial constants $c(\beta,\gamma,\delta) \in (0,\infty)$. Here the commutator $[\partial^\gamma_{(t,\Re x,\Im x,\eta)} B, \mathscr{I}]$ has the form ($\rho = (y,\eta)$)

$$\sum [\partial^{\gamma'}_{(t,\Re\rho,\Im\rho)} B, \mathscr{I}] \cdot \big(\text{Polynomial in the partial derivatives of } k\big)\,.$$

We know from Corollary 3.4 and (3.28) that

$$\|\,[\mathscr{I}, \partial^{\gamma'}_{(t,\Re\rho,\Im\rho)} B]\,\| \leqslant C_{N,K,T,\gamma'}\,\Gamma(t,x,\eta)^N.$$

□

In order to show that the formula (3.25) defines a solution of (3.1) we adapt a standard proof from classical mechanics and compare the differential of $Z_\pm$ with the pull-back under the map

$$\Theta_\pm(t,y,\eta) := \big(t, \kappa^\pm_t(y,\eta)\big) = \big(t, Q^\pm(t,y,\eta), \Xi^\pm(t,y,\eta)\big)\,, \qquad (t,y,\eta) \in \mathcal{D}(\kappa^\pm)\,,$$

of the Cartan form,

$$\omega_\pm := \xi\, dx - a_\pm(x,\xi)\, dt\,.$$

$\omega_\pm$ is considered as a form on $\mathbb{R} \times \mathbb{C}^{2d}$, so that $dx_j = d\Re x_j + i d\Im x_j$, and we abbreviate $\xi\, dx := \xi_1\, dx_1 + \cdots + \xi_d\, dx_d$, etc.

Lemma 3.8. *(i) On every compact subset $K \subset \mathcal{D}(\kappa^\pm)$ such that $|t| \leqslant t_0$ on K we have, for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^{4d+1}$,*

$$\partial^\alpha_{(t,\Re y,\Im y,\Re\eta,\Im\eta)}\big(dZ_\pm - \Theta^*_\pm\omega_\pm - y\, d\eta\big) = \mathcal{O}\big(\max_{|r|\leqslant t_0} |\Im\kappa^\pm_r(y,\eta)|^N\big)\,. \tag{3.31}$$

(ii) Let $\widetilde{\mathscr{N}_\pm}$ be the set appearing in Lemma 3.6 (so that η is real in the following). Then

$$\partial^\alpha_{(t,\Re y,\Im y,\eta)}\big(dZ_\pm - \Theta^*_\pm\omega_\pm - y\, d\eta\big) = t\,\mathcal{O}\big(\widetilde{\Gamma}^N_\pm\big)$$

on $\widetilde{\mathscr{N}_\pm}$ and for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^{3d+1}$, where the $\mathcal{O}$-symbols are uniform on compact subsets of $\widetilde{\mathscr{N}_\pm}$.

Proof. We drop all $\pm$-indices, set $\lambda := dZ - \Theta^*\omega$, and use

$$\partial_t Z = \langle\, \Xi \,|\, \nabla_\xi a(Q,\Xi)\,\rangle - a(Q,\Xi) = \langle\, \Xi \,|\, \partial_t Q\,\rangle - a(Q,\Xi) \tag{3.32}$$

to obtain

$$\lambda = (Z'_y - \langle\, \Xi \,|\, Q'_y\,\rangle)\, dy + (Z'_{\overline{y}} - \langle\, \Xi \,|\, Q'_{\overline{y}}\,\rangle)\, d\overline{y} + (Z'_\eta - \langle\, \Xi \,|\, Q'_\eta\,\rangle)\, d\eta$$
$$+ (Z'_{\overline{\eta}} - \langle\, \Xi \,|\, Q'_{\overline{\eta}}\,\rangle)\, d\overline{\eta}\,,$$

where $\langle\, \Xi \,|\, Q'_y \,\rangle\, dy$ abbreviates $\langle\, \Xi \,|\, \partial_{y_1} Q \,\rangle\, dy_1 + \cdots + \langle\, \Xi \,|\, \partial_{y_d} Q \,\rangle\, dy_d$, etc. Now, let $\varkappa$ be one of the variables $y_j, \overline{y}_j, \eta_j, \overline{\eta}_j$, $j = 1, \ldots, d$, and set $\lambda_\varkappa := Z'_\varkappa - \langle\, \Xi \,|\, Q'_\varkappa \,\rangle$. Using successively (3.32), the Hamiltonian equations (3.6), and the almost analyticity of a we find

$$\begin{aligned}
\partial_t \lambda_\varkappa \;&=\; \partial_\varkappa\big(\langle\, \Xi \,|\, \partial_t Q \,\rangle - a(Q, \Xi)\big) - \langle\, \partial_t \Xi \,|\, \partial_\varkappa Q \,\rangle - \langle\, \Xi \,|\, \partial_t \partial_\varkappa Q \,\rangle \\
&=\; \langle\, \partial_\varkappa \Xi \,|\, \nabla_\xi a(Q, \Xi) \,\rangle - \partial_\varkappa\big(a(Q, \xi)\big) + \langle\, \nabla_x a(Q, \Xi) \,|\, \partial_\varkappa Q \,\rangle \\
&=\; -\langle\, \nabla_{\overline{x}} a(Q, \Xi) \,|\, \overline{\partial_\varkappa Q} \,\rangle - \langle\, \nabla_{\overline{\xi}} a(Q, \Xi) \,|\, \overline{\partial_\varkappa \Xi} \,\rangle\,.
\end{aligned}$$

Thus, by virtue of (3.3), $\partial_t \lambda_\varkappa = G_\varkappa$, where

$$\partial^\alpha_{(t, \Re y, \Im y, \Re \eta, \Im \eta)} G_\varkappa \;=\; \mathcal{O}\big(|\Im(Q, \Xi)|^N\big)\,. \tag{3.33}$$

The initial condition $(Z, Q, \Xi)|_{t=0} = (\langle\, y \,|\, \eta \,\rangle, y, \eta)$ implies

$$\begin{aligned}
&Z'_y|_{t=0} = \eta\,, \qquad && Z'_\eta|_{t=0} = y\,, \qquad && Z'_{\overline{y}}|_{t=0} = Z'_{\overline{\eta}}|_{t=0} = 0\,, \\
&Q'_y|_{t=0} \;=\; \mathbb{1}\,, && Q'_\eta|_{t=0} = Q'_{\overline{y}}|_{t=0} = Q'_{\overline{\eta}}|_{t=0} = 0\,, &&
\end{aligned}$$

and we conclude from $\partial_t \lambda_\varkappa = G_\varkappa$ that

$$\lambda(t, y, \eta) - y\, d\eta = \sum_\varkappa \int_0^t G_\varkappa(r, y, \eta)\, dr\,, \qquad (t, y, \eta) \in \mathcal{D}(\kappa)\,,$$

which together with (3.33) yields (i). Part (ii) now follows from Lemma 3.6. $\square$

Since we have

$$Z_\pm \circ K_\pm = \psi_\pm\,, \quad Q^\pm \circ K_\pm = x\,, \quad \Xi^\pm \circ K_\pm = g^\pm\,, \tag{3.34}$$

on $\mathscr{G}^\pm$ we arrive at the following result:

Proposition 3.9. *Let $\psi_\pm$ be defined by* (3.25). *Then*

$$\partial^\alpha_{(t, \Re x, \Im x, \Re \eta)}\big(\partial_t \psi_\pm + a_\pm(x, \nabla_x \psi_\pm)\big) \;=\; \mathcal{O}(\Gamma_\pm^N)\,, \tag{3.35}$$

$$\partial^\alpha_{(t, \Re x, \Im x, \Re \eta)}(\nabla_x \psi_\pm - g^\pm) \;=\; \mathcal{O}(\Gamma_\pm^N)\,, \tag{3.36}$$

$$\partial^\alpha_{(t, \Re x, \Im x, \Re \eta)}(\nabla_\eta \psi_\pm - k^\pm) \;=\; \mathcal{O}(\Gamma_\pm^N)\,, \tag{3.37}$$

$$\partial^\alpha_{(t, \Re x, \Im x, \Re \eta)} \nabla_{\overline{x}} \psi_\pm \;=\; \mathcal{O}(\Gamma_\pm^N)\,, \tag{3.38}$$

$$\partial^\alpha_{(t, \Re x, \Im x, \Re \eta)} \nabla_{\overline{\eta}} \psi_\pm \;=\; \mathcal{O}(\Gamma_\pm^N)\,, \tag{3.39}$$

on $\mathscr{N}_\pm$, for $N \in \mathbb{N}$ and every multi-index $\alpha \in \mathbb{N}_0^{3d+1}$. All $\mathcal{O}$-symbols are uniform on compact subsets of $\mathscr{N}_\pm$.

Proof. Lemma 3.8(ii) and $\Gamma = \widetilde{\Gamma} \circ K$ imply

$$\partial^\alpha_{(t, \Re x, \Im x, \Re \eta)} K_\pm^*\big(dZ_\pm - \Theta_\pm^* \omega_\pm - y\, d\eta\big) \;=\; \mathcal{O}(\Gamma_\pm^N)$$

on $K^{-1}(\widetilde{\mathscr{N}_\pm}) = \mathscr{N}_\pm$. On the other hand (3.34) shows that, on $\mathscr{G}^\pm$,

$$\begin{aligned} K_\pm^*\big(dZ_\pm - \Theta_\pm^*\omega_\pm - y\,d\eta\big) \;&=\; d(K_\pm^* Z_\pm) - (\Theta_\pm \circ K_\pm)^*\omega_\pm - k^\pm\,d\eta \\ &=\; d\psi_\pm - g^\pm\,dx + a_\pm(x, g^\pm)\,dt - k^\pm\,d\eta\,. \end{aligned}$$

□

The last corollary of this section summarizes the properties of $\psi_\pm$ on the real domain, where the weight $\Gamma_\pm$ can actually be replaced by $\Im\psi_\pm$, so that we arrive at the desired solution of the problem (3.1)&(3.2).

Corollary 3.10. *(i) There is some real neighborhood, $\mathscr{M}_{\mathbb{R}}^\pm$, of $\mathscr{E}^\pm$ in $\mathbb{R}_0^+ \times \mathbb{R}^{2d}$ such that, for all $(t,x,\eta) \in \mathscr{M}_{\mathbb{R}}^\pm$,*

$$\Im\psi_\pm(t,x,\eta) \;\geqslant\; \frac{1}{\mathcal{O}(1)}|\,\Im g^\pm(t,x,\eta)|^2, \tag{3.40}$$

$$\Im\psi_+(t,x,\eta) \;=\; 0 \quad\Leftrightarrow\quad (t,x,\eta) \in \mathscr{E}^+, \tag{3.41}$$

$$\Im\psi_-(t,x,\eta) \;=\; 0 \quad\Leftrightarrow\quad t = 0\,,\; x,\eta \in \mathbb{R}^d. \tag{3.42}$$

Consequently,

$$\Im\psi_\pm \;\geqslant\; \frac{1}{\mathcal{O}(1)}\,\Gamma_\pm \qquad \textit{on } \mathscr{M}_{\mathbb{R}}^\pm\,, \tag{3.43}$$

so that (3.35)–(3.39) *hold true on $\mathscr{M}_{\mathbb{R}}^\pm$ with the right hand sides replaced by $\mathcal{O}_N((\Im\psi_\pm)^N)$. In particular,*

$$\partial^\alpha_{(t,\Re x,\Im x,\Re\eta)}\big(\partial_t\psi_\pm + a_\pm(x,\nabla_x\psi_\pm)\big) \;=\; \mathcal{O}\big((\Im\psi_\pm)^N\big) \qquad \textit{on } \mathscr{M}_{\mathbb{R}}^\pm\,. \tag{3.44}$$

(All $\mathcal{O}$-symbols are uniform on compact subsets of $\mathscr{M}_{\mathbb{R}}^\pm$.)
(ii) For all $(t,x,0) \in \mathscr{D}^+$ and $\beta \in \mathbb{N}_0^{d+1}$,

$$\partial^\beta_{(t,x)}\psi_+(t,x,0) = 0\,, \qquad \nabla_\eta\psi_+(t,x,0) = X(-t,x,\nabla\varphi(x))\,. \tag{3.45}$$

Proof. (i): On account of (3.27) and (3.34),

$$\Im\psi_\pm - \langle\,\Im x\,|\,\Re g^\pm\,\rangle \;\geqslant\; -\mathcal{O}(1)\,|\Im(x,g^\pm)|^3 \qquad \text{on } \mathscr{N}_\pm\,. \tag{3.46}$$

We recall that $\mathscr{N}_\pm$ is a neighborhood of $\mathscr{E}^\pm$ in $\mathbb{R}_0^+ \times \mathbb{C}^d \times \mathbb{R}^d$ and, hence, $(\mathscr{N}_\pm)_{\mathbb{R}} := \mathscr{N}_\pm \cap (\mathbb{R}_0^+ \times \mathbb{R}^{2d})$ is a neighborhood of $\mathscr{E}^\pm$ in $\mathbb{R}_0^+ \times \mathbb{R}^{2d}$. Now, let $(t_0,x_0,\eta_0) \in \mathscr{E}^\pm$ and let $K \subset (\mathscr{N}_\pm)_{\mathbb{R}}$ be a compact neighborhood of (t_0,x_0,η_0) in $(\mathscr{N}_\pm)_{\mathbb{R}}$. By choosing $\varepsilon_0 > 0$ sufficiently small we can ensure that $(t, x + \varepsilon\,\Im g^\pm(t,x,\eta),\eta) \in K' \subset (\mathscr{N}_\pm)_{\mathbb{R}}$, for every $(t,x,\eta) \in K$ and $|\varepsilon| < \varepsilon_0$, where K' is compact, too. According to (3.46) there exist $C, C' \in (0,\infty)$ such that, for all $(t,x,\eta) \in K$,

$$\begin{aligned} \Im\psi_\pm(t, x - \varepsilon\,\Im g^\pm(t,x,\eta),\eta) \;&\geqslant\; -C\,\big|\Im g^\pm\big(t, x - \varepsilon\,\Im g^\pm(t,x,\eta),\eta\big)\big|^3 \\ &\geqslant\; -C'\,\big|\Im g^\pm(t,x,\eta)\big|^3\,. \end{aligned}$$

Taylor expanding the left hand side of the previous estimate with respect to x using (3.36) and (3.38) we obtain

$$\Im\psi_\pm \geqslant \varepsilon\,|\Im g^\pm|^2 - C''\left(|\Im g^\pm|^3 + \varepsilon^2\,|\Im g^\pm|^2\right) - \varepsilon\,C_{N_0}\,|\Im\psi_\pm|^{N_0} \qquad \text{on } K\,,$$

for some $N_0 \in \mathbb{N}$, $N_0 \geqslant 2$, and $C'', C_{N_0} \in (0,\infty)$. Now, we choose $\varepsilon \in (0, \frac{1}{2C''})$, $\varepsilon < \varepsilon_0$, such that $\varepsilon\,C_{N_0}\,|\Im\psi_\pm|^{N_0-1} < 1/2$ on K. Then $\Im\psi_\pm + (1/2)|\Im\psi_\pm| \geqslant (\varepsilon/2)|\Im g^\pm|^2 - C''|\Im g^\pm|^3$ on K. Next, we recall from (3.21) and (3.22) that $\Im g^\pm = 0$ on $\mathscr{E}^\pm$. Therefore, we may further ensure that $C''|\Im g| < \varepsilon/4$ on K by possibly restricting the compact neighborhood K of (t_0, x_0, η_0) suitably and we obtain (3.40). Finally, the set $\mathscr{M}^\pm_{\mathbb{R}}$ is defined as the union of all sets $\mathring{K}$ obtained as above for every $(t_0, x_0, \eta_0) \in \mathscr{E}^\pm$.

Next, we prove (3.41) and (3.42). First, let $(t, x, \eta) \in \mathscr{E}^\pm$. At $t = 0$ we have $\Im\psi_\pm(0, x, \eta) = \Im\langle\, x \,|\, \eta\,\rangle = 0$. If $(t, x, 0) \in \mathscr{D}^+$, then we know that $\kappa(r, k(t, x, 0), 0) \in K_0 \times \{0\}$, for all $r \in [0, t]$, whence $\mathscr{A}_+(\kappa(r, k(t, x, 0), 0)) = 0$, $r \in [0, t]$, due to (3.4). Recalling the definition (3.25) of ψ_+ we see that $\psi_+(t, x, 0) = 0$, which also proves the first assertion of (ii).

Conversely, assume that $(t, x, \eta) \in \mathscr{M}^\pm_{\mathbb{R}}$ with $t > 0$ and $\Im\psi_\pm(t, x, \eta) = 0$. Then (3.40) implies that $\Im g^\pm(t, x, \eta) = 0$ and (3.28) and (3.34) show that $(y, \eta) := (k^\pm(t, x, \eta), \eta)$ and $(x, g^\pm(t, x, \eta))$ are connected by a purely real integral curve of $\widehat{\mathscr{H}_{a_\pm}}$. Then $\Im a_\pm(y, \eta) < 0$ implies $\Im\psi_\pm(t, x, \eta) > 0$ on account of (2.17), (2.25), and (3.27). In the minus-case we thus get a contradiction to (2.25) showing that there is no $(t, x, \eta) \in \mathscr{M}^-_{\mathbb{R}}$ with $t > 0$ and $\Im\psi_-(t, x, \eta) = 0$. In the plus-case it follows that $\Im a_+(y, \eta) = 0$, that is, $y \in K_0$ and $\eta = 0$ by (2.18). Lemma 3.2 implies that $(t, y, 0) \in \widetilde{\mathscr{D}}^+$, thus $(t, x, 0) \in \mathscr{D}^+$.

Finally, Part (ii) follows from (2.18), (3.19), (3.21), and (3.35)–(3.37). $\square$

4. The transport equations

4.1. **Formal ansatz for a parametrix.** In order to construct a parametrix for the conjugated Dirac operator $D_{h,V,\varphi}$ we split, roughly speaking, $D_{h,V,\varphi}$ micro-locally into a plus and a minus part by means of the projections introduced in (1.12). For each of these parts, again roughly speaking, we construct parametrices for the corresponding "heat equations" (backwards in time in the minus case) and integrate the latter with respect to the time variable. The parametrices for the heat equations are obtained as Fourier integral operators with complex-valued phase functions. More precisely, our ansatz for the Green kernel reads

$$\begin{aligned} D^{-1}_{h,V,\varphi}(x, y) = \sum_{\sharp\in\{+,-\}} \sharp \int_0^\infty \int_{\mathbb{R}^d} e^{i\psi_\sharp(t,x,\eta)/h - i\langle\, y \,|\, \eta\,\rangle/h} \sum_{\nu=0}^\infty h^\nu\, B^\nu_\sharp(t, x, \eta)\, \frac{d\eta\, dt}{(2\pi h)^d\, h} \\ + \check{q}(x, x - y)\,. \end{aligned}$$

The symbol q additionally appearing here accounts for the elliptic part of $D_{h,V,\varphi}$ and is constructed in Section 5 below. To find equations determining $\psi_\pm$ and $B^\nu_\pm$ we calculate formally

$$
\begin{aligned}
&e^{-i\psi_\pm/h}\,\big(\pm h\,\partial_t + \boldsymbol{\alpha}\cdot(-ih\,\nabla + i\nabla\varphi) + \alpha_0 + V\big)e^{i\psi_\pm/h}\sum_{\nu=0}^{\infty} h^\nu\, B^\nu_\pm \\
&\quad= \big(\pm i\,\partial_t\psi_\pm + \boldsymbol{\alpha}\cdot(\nabla_x\psi_\pm + i\nabla\varphi) + \alpha_0 + V\big)\, B^0_\pm \\
&\qquad + \sum_{\nu=1}^{\infty} h^\nu \Big\{(\pm\partial_t - i\boldsymbol{\alpha}\cdot\nabla)B^{\nu-1}_\pm \\
&\qquad\qquad + \big(\pm i\partial_t\psi_\pm + \boldsymbol{\alpha}\cdot(\nabla_x\psi_\pm + i\nabla\varphi) + \alpha_0 + V\big)\, B^\nu_\pm\Big\} \overset{!}{=} 0\,.
\end{aligned}
\tag{4.1}
$$

In the sequel we fix a smooth cut-off function, $\chi \in C_0^\infty(\mathbb{C}^{2d})$, such that $\chi \equiv 1$ on some small real neighborhood of $K_0 \times \{0\}$, $0 \leqslant \chi \leqslant 1$ on $\mathbb{R}^{2d}$, and such that $\mathrm{supp}(\chi)$ is contained in some small complex neighborhood of $K_0 \times \{0\}$. We assume that χ is an almost analytic extension of $\chi\!\restriction_{\mathbb{R}^{2d}}$, so that

$$
\big|\partial^\alpha_{(\Re y,\Im y,\Re\eta,\Im\eta)}\nabla_{(\overline{y},\overline{\eta})}\chi(y,\eta)\big| \leqslant C_{N,\alpha}\,|\Im(y,\eta)|^N, \qquad (y,\eta)\in\mathbb{C}^{2d},
$$

for all $N\in\mathbb{N}$, $\alpha\in\mathbb{N}_0^{4d}$, and suitable constants $C_{N,\alpha}\in(0,\infty)$.

Let us suppose for the moment that the matrix-valued amplitudes $B^0_\pm$ satisfy

$$
\begin{aligned}
(\mathbf{T_0}):\quad &\partial^\alpha_{(t,\Re x,\Im x,\eta)}\big\{B^0_\pm(t,x,\eta) - \Lambda^\pm\big(x,\nabla_x\psi_\pm(t,x,\eta) + i\nabla\varphi(x)\big)\,B^0_\pm(t,x,\eta)\big\} \\
&\qquad = \mathcal{O}(\Gamma^N_\pm)\,, \\
&B^0_\pm(0,x,\eta) = \chi(x,\eta)\,\Lambda^\pm\big(x,\eta + i\nabla\varphi(x)\big)\,.
\end{aligned}
$$

From (4.1) we further obtain the transport equations

$$
\begin{aligned}
(\mathbf{T_\nu})_{\boldsymbol{\nu}\geqslant\mathbf{1}}:\quad &\partial^\alpha_{(t,\Re x,\Im x,\eta)}\big\{(\mp\partial_t + i\boldsymbol{\alpha}\cdot\nabla)\,B^{\nu-1}_\pm \\
&\qquad - (\pm i\partial_t\psi_\pm + \boldsymbol{\alpha}\cdot(\nabla_x\psi_\pm + i\nabla\varphi) + \alpha_0 + V)\,B^\nu_\pm\big\} = \mathcal{O}_N(\Gamma^N_\pm)\,, \\
&B^\nu_+|_{t=0} = -B^\nu_-|_{t=0}\,.
\end{aligned}
$$

If $(\mathbf{T_0})$ is fulfilled, then the matrix in front of $B^0_\pm$ in (4.1) can be replaced by one of its eigenvalues and we find the eikonal equations

$$
\pm i\,\partial_t\psi_\pm \pm \sqrt{1 + (\nabla_x\psi_\pm + i\nabla\varphi)^2} + V = \mathcal{O}(\Gamma^N_\pm)\,,
$$

which, according to the definition (2.16), are equivalent to the problems (3.1) solved in Section 3. Again the error terms $\mathcal{O}(\Gamma^N_\pm)$ in the transport equations cannot be avoided because the transport equations are complex-valued. They do, however, not destroy the WKB construction as we shall see later on in Proposition 4.8.

4.2. **Solution of the transport equations.** In the rest of Section 4 we assume that V fulfills Hypothesis 1.1, $x_\star$ and $y_\star$ fulfill Hypothesis 1.2, and that φ and $\psi_\pm$ are the functions given by Propositions 2.1 and 2.2 and by (3.25), respectively. All $\mathcal{O}$-symbols are uniform on compact subsets and the variable η will be real.

The transport equations $(\mathbf{T_0}), (\mathbf{T_1}), \ldots$ are solved by means of a strategy we learned from [11]. Following this strategy we have, however, to keep track of the error terms and to put some factors i and some additional minus signs in the right places. For we consider a "heat equation" with different time directions for the plus and minus parts of the symbol rather than the usual Dirac equation whose scattering theory is discussed in [11]. For the convenience of the reader it thus makes sense to give a self-contained discussion of the transport equations. As a first step we introduce gamma-matrices

$$\gamma_0 := \alpha_0\,, \quad \gamma_j := -\alpha_0\,\alpha_j\,, \quad j = 1, \ldots, d\,,$$

so that

$$(\gamma_0)^2 = \mathbb{1}\,, \qquad (\gamma_j)^2 = -\mathbb{1}\,, \quad j = 1, \ldots, d\,, \tag{4.2}$$

$$\{\gamma_\mu\,,\,\gamma_\nu\} = 0\,, \qquad 0 \leqslant \mu < \nu \leqslant d\,. \tag{4.3}$$

Furthermore, we set

$$\begin{aligned}
&\partial_0^\pm := \pm i\partial_t\,, \qquad \partial_j^\pm := -\partial_{x_j}\,,\\
&\Pi_0^\pm := \pm\sqrt{1 + (\nabla_x\psi_\pm + i\nabla\varphi)^2}\,, \quad \Pi_j^\pm := \partial_{x_j}\psi_\pm + i\partial_{x_j}\varphi\,,\\
&\mathbf{\Pi}^\pm := \big(\Pi_1^\pm, \ldots, \Pi_d^\pm\big)\,,
\end{aligned}$$

where $j = 1, \ldots, d$, and

$$\tilde\partial^\pm := \sum_{\mu=0}^{d} \gamma_\mu\,\partial_\mu^\pm\,, \qquad \tilde\Pi^\pm := \sum_{\mu=0}^{d} \gamma_\mu\,\Pi_\mu^\pm \quad \text{on } \mathscr{N}_\pm\,.$$

We recall that the sets $\mathscr{N}_\pm$ have been introduced in (3.29). We transform the transport equations into a new sequence of equations on $\mathscr{N}_\pm$ given as

$$(\mathbf{K}_\nu)\begin{cases}\partial^\alpha_{(t,\Re x,\Im x,\eta)}\big((\tilde\Pi^\pm - 1)\,B_\pm^\nu + i\tilde\partial^\pm B_\pm^{\nu-1}\big) \;=\; \mathcal{O}_N(\Gamma_\pm^N)\,, \quad N \in \mathbb{N}\,,\\ \partial^\alpha_{(t,\Re x,\Im x,\eta)}(\tilde\Pi^\pm + 1)\,\tilde\partial^\pm\,B_\pm^\nu \;=\; \mathcal{O}_N(\Gamma_\pm^N)\,, \quad N \in \mathbb{N}\,,\end{cases}$$

where $\nu \in \mathbb{N}_0$ and $B_\pm^{-1} := 0$.

Lemma 4.1. *If $B_\pm^\nu$, $\nu \in \mathbb{N}_0$, satisfy the first equation in $(\mathbf{K_1}), (\mathbf{K_2}), \ldots$ on some neighborhood, $\mathscr{N}_\pm' \subset \mathscr{N}_\pm$, of $\mathscr{E}_\pm$ in $\mathbb{R}_0^+ \times \mathbb{C}^d \times \mathbb{R}^d$, then they satisfy the transport equations $(\mathbf{T_1}), (\mathbf{T_2}), \ldots$ on $\mathscr{N}_\pm'$, too.*

Proof. Let $\nu \in \mathbb{N}$. Multiplying the first equation in $(\mathbf{K}_\nu)$ with α_0 and using $\alpha_0^2 = \mathbb{1}$ we obtain

$$\begin{aligned}\partial^\alpha_{(t,\Re x,\Im x,\eta)}\big\{(\mp\partial_t + i\boldsymbol{\alpha}\cdot\nabla)\, B_\pm^{\nu-1} + \big(\Pi_0^\pm + V \\ - \boldsymbol{\alpha}\cdot(\nabla_x\psi_\pm + i\nabla\varphi) - \alpha_0 - V\big)\, B_\pm^\nu\big\} = \mathcal{O}_N(\Gamma_\pm^N)\,. \end{aligned}\tag{4.4}$$

In view of the Hamilton-Jacobi equation $\partial^\alpha_{(t,\Re x,\Im x,\eta)}(\pm i\partial_t\psi_\pm + \Pi_0^\pm + V) = \mathcal{O}_N(\Gamma_\pm^N)$, $N\in\mathbb{N}$, we see that (4.4) is equivalent to $(\mathbf{T}_{\boldsymbol{\nu}})$. ☐

In what follows we abbreviate

$$\sigma_{\mu\nu} \,:=\, \frac{i}{2}\,[\gamma_\mu\,,\,\gamma_\nu] \,=\, i\gamma_\mu\,\gamma_\nu\,,\quad (\mathrm{rot}^\pm\,\Pi^\pm)_{\mu\nu} \,:=\, (\partial_\mu^\pm\Pi_\nu^\pm) - (\partial_\nu^\pm\Pi_\mu^\pm)\,,\tag{4.5}$$

for all $\mu,\nu = 0,\dots,d$, $\mu\neq\nu$. We introduce the following matrix-valued partial differential operator on $\mathscr{N}_\pm$,

$$L^\pm := \pm i\{\partial_t,\Pi_0^\pm\} + \sum_{j=1}^d\{\partial_{x_j},\Pi_j^\pm\} - i\sum_{0\leqslant\mu<\nu\leqslant d}\sigma_{\mu\nu}\,(\mathrm{rot}^\pm\,\Pi^\pm)_{\mu\nu}\,.\tag{4.6}$$

On account of (4.2), (4.3), and (4.5) we can re-write $L^\pm$ as

$$L^\pm \,=\, \sum_{\mu=0}^d\gamma_\mu^2\,\{\partial_\mu^\pm,\Pi_\mu^\pm\} + \sum_{\substack{\mu,\nu=0\\ \mu\neq\nu}}^d\gamma_\mu\,\gamma_\nu\,(\partial_\mu^\pm\circ\Pi_\nu^\pm + \Pi_\mu^\pm\,\partial_\nu^\pm) \,=\, \{\tilde\partial^\pm\,,\,\tilde\Pi^\pm\}\,.\tag{4.7}$$

Lemma 4.2. *Let $\mathscr{N}_\pm$ be the neighborhood of $\mathscr{E}_\pm$ in $\mathbb{R}_0^+\times\mathbb{C}^d\times\mathbb{R}^d$ introduced in* (3.29) *and suppose that the matrix-valued function $R\in C^\infty(\mathscr{N}_\pm,\mathscr{L}(\mathbb{C}^{d_*}))$ satisfies*

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}d_{\overline{x}}R \,=\, \mathcal{O}(\Gamma_\pm^N)\,,\qquad N\in\mathbb{N}\,,\ \alpha\in\mathbb{N}_0^{3d+1}.\tag{4.8}$$

Furthermore, let $c\in C^\infty(\mathbb{C}^d\times\mathbb{R}^d)$ satisfy

$$\partial^\alpha_{(\Re x,\Im x,\eta)}d_{\overline{x}}c \,=\, \mathcal{O}(|\Im x|^N)\,,\qquad N\in\mathbb{N}\,,\ \alpha\in\mathbb{N}_0^{3d}.\tag{4.9}$$

Then there exist $B^\pm\in C^\infty(\mathscr{N}_\pm,\mathscr{L}(\mathbb{C}^{d_*}))$ such that

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}(L^\pm\,B^\pm - R) = \mathcal{O}(\Gamma_\pm^N),\ \ \partial^\beta_{(\Re x,\Im x,\eta)}(B^\pm|_{t=0} - c) = \mathcal{O}(|\Im x|^N),\tag{4.10}$$

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}d_{\overline{x}}B^\pm = \mathcal{O}(\Gamma_\pm^N),\tag{4.11}$$

for all $N\in\mathbb{N}$, $\alpha\in\mathbb{N}_0^{3d+1}$, and $\beta\in\mathbb{N}_0^{3d}$. If $C^\pm\in C^\infty(\mathscr{N}_\pm,\mathscr{L}(\mathbb{C}^{d_*}))$ is another solution of (4.10) *and* (4.11), *then $\partial^\alpha_{(t,\Re x,\Im x,\eta)}(B^\pm - C^\pm) = \mathcal{O}(\Gamma_\pm^N)$, for all $N\in\mathbb{N}$ and $\alpha\in\mathbb{N}_0^{3d+1}$.*

Proof. By definition we have

$$L^\pm = \pm 2i\Pi_0^\pm\,\partial_t + 2\mathbf{\Pi}^\pm\cdot\nabla_x + M^\pm\,,$$

where

$$M^{\pm} := \pm i(\partial_t \Pi_0^{\pm}) + \operatorname{div}_x \mathbf{\Pi}^{\pm} - i \sum_{0 \leqslant \mu < \nu \leqslant d} \sigma_{\mu\nu} \, (\operatorname{rot}^{\pm} \Pi^{\pm})_{\mu\nu} \, .$$

We further set

$$W^{\pm} := -(\pm 2i\Pi_0^{\pm})^{-1} \, M^{\pm} \, , \qquad S^{\pm} := (\pm 2i\Pi_0^{\pm})^{-1} \, R \, ,$$

and consider the maximal solutions, $\widetilde{B}^{\pm}$, of the ordinary differential equations

$$\partial_t \widetilde{B}^{\pm}(t,y,\eta) = W^{\pm}\big(t, Q^{\pm}(t,y,\eta),\eta\big) \, \widetilde{B}^{\pm}(t,y,\eta) + S^{\pm}\big(t, Q^{\pm}(t,y,\eta),\eta\big) \tag{4.12}$$

defined for $(t,y,\eta) \in \widetilde{\mathscr{N}_{\pm}'}$ with boundary condition $\widetilde{B}(0,y,\eta) = c(y,\eta)$. We further introduce the vector field

$$Z_{\pm} := \nabla_{\xi} a_{\pm}(x, g^{\pm}) \cdot \nabla_x$$

on $\mathscr{G}^{\pm}$. Then an appropriate restriction of $Q^{\pm} : \mathcal{D}(\kappa^{\pm}) \to \mathbb{C}^d$ is equal to the flow of the real vector field $\widehat{Z}_{\pm} := Z_{\pm} + \overline{Z}_{\pm}$, because $g^{\pm}(t, Q^{\pm}(t,y,\eta),\eta) = \Xi^{\pm}(t,y,\eta)$, for all $(t,y,\eta) \in \mathcal{D}(\kappa^{\pm})$ such that $(t, Q^{\pm}(t,y,\eta),\eta) \in \mathscr{G}^{\pm}$. Setting

$$B^{\pm}(t,x,\eta) := \widetilde{B}^{\pm}\big(t, k^{\pm}(t,x,\eta),\eta\big)$$

we thus have

$$\begin{aligned}
(\partial_t B^{\pm} + \widehat{Z}_{\pm} \, B^{\pm})\big(t, Q^{\pm}(t,y,\eta),\eta\big) &= \frac{d}{dt} B^{\pm}\big(t, Q^{\pm}(t,y,\eta),\eta\big) \\
&= W^{\pm}\big(t, Q^{\pm}(t,y,\eta),\eta\big) \, \widetilde{B}^{\pm}(t,y,\eta) + S^{\pm}\big(t, Q^{\pm}(t,y,\eta),\eta\big) \, ,
\end{aligned}$$

because $k^{\pm}(t, Q^{\pm}(t,y,\eta),\eta) = y$. In view of $(\pm 2i\Pi_0^{\pm})^{-1} 2\mathbf{\Pi}^{\pm} = \nabla_{\xi} a_{\pm}(x, \nabla_x \psi_{\pm})$ it follows that

$$\begin{aligned}
&(\pm 2i\Pi_0^{\pm})^{-1}(L^{\pm} \, B^{\pm} - R) \\
&\quad = \big(\partial_t + \nabla_{\xi} a_{\pm}(x, \nabla_x \psi_{\pm}) \cdot \nabla_x - W^{\pm}\big) \, B^{\pm} - S^{\pm} \\
&\quad = -\overline{\nabla_{\xi} a_{\pm}(x, g^{\pm})} \cdot \nabla_{\overline{x}} B^{\pm} + \big(\nabla_{\xi} a_{\pm}(x, \nabla_x \psi_{\pm}) - \nabla_{\xi} a_{\pm}(x, g^{\pm})\big) \cdot \nabla_x B^{\pm}.
\end{aligned} \tag{4.13}$$

From now on we drop all sub- and superscripts $\pm$ in the existence part of this proof since they do not play any role anymore. By virtue of (3.36) we know that $\partial^{\alpha}_{(t,\Re x,\Im x,\eta)}(\nabla_x \psi - g) = \mathcal{O}(\Gamma^N)$, whence

$$\partial^{\alpha}_{(t,\Re x,\Im x,\eta)}\big(\nabla_{\xi} a(x, \nabla_x \psi) - \nabla_{\xi} a(x,g)\big) = \mathcal{O}(\Gamma^N) \, , \tag{4.14}$$

for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^{3d+1}$. Furthermore, it is clear that all partial derivatives of B are bounded on compact subsets of $\mathscr{N}$. To study $d_{\overline{x}} B$ we write

$$d_{\overline{x}} B = (d_y \widetilde{B})(t,k,\eta) \, d_{\overline{x}} k + (d_{\overline{y}} \widetilde{B})(t,k,\eta) \, d_{\overline{x}} \overline{k} \, .$$

Here we know from Lemma 3.7 that $\partial^\alpha_{(t,\Re x,\Im x,\eta)} d_{\overline{x}} k = \mathcal{O}(\Gamma^N)$ and it suffices to find a similar bound on $d_{\overline{y}}\widetilde{B}$. To this end we differentiate the differential equation (4.12) to get

$$\begin{aligned}\partial_t d_{\overline{y}}\widetilde{B} &= W\big(t,Q,\eta\big)\, d_{\overline{y}}\widetilde{B} + \big(d_x W\big(t,Q,\eta\big)\, d_{\overline{y}}Q\big)\,\widetilde{B} + \big(d_{\overline{x}} W\big(t,Q,\eta\big)\, d_{\overline{y}}\overline{Q}\big)\,\widetilde{B} \\ &\quad + d_x S\big(t,Q,\eta\big)\, d_{\overline{y}}Q + d_{\overline{x}} S\big(t,Q,\eta\big)\, d_{\overline{y}}\overline{Q}\,,\end{aligned}$$

which on account of (3.15), (3.38), (4.8), and $|\Im Q|^2 \leqslant \mathcal{O}(1)\widetilde{\Gamma}$, shows that

$$\partial^\alpha_{(t,\Re y,\Im y,\eta)}\big(\partial_t d_{\overline{y}}\widetilde{B} - W\big(t,Q,\eta\big)\, d_{\overline{y}}\widetilde{B}\big) \;=\; \mathcal{O}(\widetilde{\Gamma}^N)\,. \tag{4.15}$$

Since $\widetilde{B}(0,y,\eta) = B(0,y,\eta)$ we also have $\partial^\alpha_{(\Re y,\Im y,\eta)} d_{\overline{y}}\widetilde{B}|_{t=0} = \partial^\alpha_{(\Re y,\Im y,\eta)} d_{\overline{y}} c = \mathcal{O}(|\Im y|^N)$. Therefore, we first obtain from Duhamel's formula and $\widetilde{\Gamma}(s,y,\eta) \leqslant \mathcal{O}(1)\,\widetilde{\Gamma}(t,y,\eta)$, $s\in[0,t]$, that

$$d_{\overline{y}}\widetilde{B} \;=\; U_{t,0}\,\mathcal{O}(|\Im y|^N) + \int_0^t U_{t,s}\,\mathcal{O}\big(\widetilde{\Gamma}(s,y,\eta)^N\big)\,ds \;=\; \mathcal{O}\big(\widetilde{\Gamma}(t,y,\eta)^N\big)\,, \tag{4.16}$$

where $U_{t,s}$ fulfills $\partial_t U_{t,s} = W(t,Q(t,y,\eta),\eta)\,U_{t,s}$, $U_{s,s} = \mathbb{1}$. Now, suppose we have shown that $\partial^\beta_{(\Re y,\Im y,\eta)} d_{\overline{y}}\widetilde{B} = \mathcal{O}\big(\widetilde{\Gamma}^N\big)$ is valid, for all $\beta\in\mathbb{N}_0^{3d}$ with $|\beta|\leqslant n\in\mathbb{N}_0$ and let $\alpha\in\mathbb{N}_0^{3d}$, $|\alpha| = n+1$. Then we obtain

$$\partial_t\partial^\alpha_{(\Re y,\Im y,\eta)} d_{\overline{y}}\widetilde{B} - W(t,Q,\eta)\,\partial^\alpha_{(\Re y,\Im y,\eta)} d_{\overline{y}}\widetilde{B} \;=\; \mathcal{O}(\widetilde{\Gamma}^N) \tag{4.17}$$

from (4.15), and we again conclude that $\partial^\alpha_{(\Re y,\Im y,\eta)} d_{\overline{y}}\widetilde{B} = \mathcal{O}\big(\widetilde{\Gamma}^N\big)$. Using the differential equation (4.15) and the usual bootstrap argument to include the time derivatives we further see that $\partial^\alpha_{(t,\Re y,\Im y,\eta)} d_{\overline{y}}\widetilde{B} = \mathcal{O}\big(\widetilde{\Gamma}^N\big)$, for all $N\in\mathbb{N}$ and $\alpha\in\mathbb{N}_0^{3d+1}$. Altogether we arrive at (4.11) and from (4.13) and (4.14) we infer that (4.10) holds true also.

Now, suppose that $C^\pm$ is another solution of (4.10) and (4.11). Then we have $(\pm 2i\Pi_0^\pm)^{-1}L^\pm(B^\pm - C^\pm) = G^\pm$, where $\partial^\alpha_{(t,\Re x,\Im x,\eta)} G^\pm = \mathcal{O}(\Gamma_\pm^N)$, or (compare (4.13))

$$\begin{aligned}&\big(\partial_t + \widehat{Z}_\pm - W^\pm\big)(B^\pm - C^\pm) \;=\; G^\pm \\ &\quad + \big\{\overline{\nabla_\xi a_\pm(x,g^\pm)}\cdot\nabla_{\overline{x}} - \big(\nabla_\xi a_\pm(x,\nabla_x\psi_\pm) - \nabla_\xi a_\pm(x,g^\pm)\big)\cdot\nabla_x\big\}(B^\pm - C^\pm)\,.\end{aligned}$$

Let $\widetilde{S}^\pm$ denote the right hand side of the previous identity. Setting $\widetilde{E}^\pm := (B^\pm - C^\pm)(t,Q^\pm,\eta)$ we then have

$$\partial_t\widetilde{E}^\pm - W^\pm(t,Q^\pm,\eta)\,\widetilde{E}^\pm \;=\; \widetilde{S}^\pm(t,Q^\pm,\eta)\,, \qquad \partial^\beta_{(\Re y,\Im y,\eta)}\widetilde{E}^\pm|_{t=0} = \mathcal{O}(|\Im y|^N)\,,$$

where $\partial^\alpha_{(t,\Re x,\Im x,\eta)}\widetilde{S}^\pm = \mathcal{O}(\Gamma_\pm^N)$. By the same induction argument as the one used above to discuss $d_{\overline{y}}\widetilde{B}$ we infer that $\partial^\alpha_{(t,\Re y,\Im y,\eta)}\widetilde{E}^\pm = \mathcal{O}(\widetilde{\Gamma}_\pm^N)$, which implies $\partial^\alpha_{(t,\Re x,\Im x,\eta)}(B^\pm - C^\pm) = \mathcal{O}(\Gamma_\pm^N)$, for $N\in\mathbb{N}$, $\alpha\in\mathbb{N}_0^{3d+1}$. $\square$

The next lemma will be applied to the choices $f^\pm = i\tilde{\partial}^\pm B_\pm^\nu$ and $u^\pm = B_\pm^{\nu+1}$.

Lemma 4.3. *Let the matrix-valued functions $f^\pm, u^\pm \in C^\infty(\mathscr{N}_\pm, \mathscr{L}(\mathbb{C}^{d_*}))$ satisfy the equations*

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}(\tilde{\Pi}^\pm + 1)\, f^\pm \;=\; \mathcal{O}(\Gamma_\pm^N), \tag{4.18}$$

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}\big(L^\pm\, u^\pm + \tilde{\partial}^\pm f^\pm\big) \;=\; \mathcal{O}(\Gamma_\pm^N), \tag{4.19}$$

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)} d_{\overline{x}} u^\pm \;=\; \mathcal{O}(\Gamma_\pm^N), \tag{4.20}$$

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)} d_{\overline{x}} f^\pm \;=\; \mathcal{O}(\Gamma_\pm^N), \tag{4.21}$$

for every $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^{3d+1}$, and

$$\partial^\beta_{(\Re x,\Im x,\eta)}\big((\tilde{\Pi}^\pm(0,x,\eta) - 1)\, u^\pm(0,x,\eta) + f^\pm(0,x,\eta)\big) \;=\; \mathcal{O}(|\Im x|^N)\,, \tag{4.22}$$

for all $(0,x,\eta) \in \mathscr{N}_\pm$, $N \in \mathbb{N}$, and $\beta \in \mathbb{N}_0^{3d}$. Then $u^\pm$ fulfills the following equations on $\mathscr{N}_\pm$, for every $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^{3d+1}$,

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}\big((\tilde{\Pi}^\pm - 1)\, u^\pm + f^\pm\big) \;=\; \mathcal{O}(\Gamma_\pm^N)\,, \tag{4.23}$$

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}(\tilde{\Pi}^\pm + 1)\, \tilde{\partial}^\pm u^\pm \;=\; \mathcal{O}(\Gamma_\pm^N)\,. \tag{4.24}$$

Proof. On account of (4.2) and (4.3) we have $(\tilde{\Pi}^\pm)^2 = (\Pi_0^\pm)^2 - (\nabla_x\psi_\pm + i\nabla\varphi)^2 = \mathbb{1}$, which together with (4.7) gives $(\tilde{\Pi}^\pm - 1)\, L^\pm = L^\pm\, (\tilde{\Pi}^\pm - 1)$. From this we infer that

$$\begin{aligned} L^\pm\, \big((\tilde{\Pi}^\pm - 1)\, u^\pm + f^\pm\big) \;&=\; (\tilde{\Pi}^\pm - 1)\, L^\pm\, u^\pm + (\tilde{\partial}^\pm\tilde{\Pi}^\pm + \tilde{\Pi}^\pm\tilde{\partial}^\pm)\, f^\pm \\ &=\; (\tilde{\Pi}^\pm - 1)\, (L^\pm\, u^\pm + \tilde{\partial}^\pm f^\pm) + \tilde{\partial}^\pm(\tilde{\Pi}^\pm + 1)\, f^\pm\,, \end{aligned}$$

thus $\partial^\alpha_{(t,\Re x,\Im x,\eta)} L^\pm\, \big((\tilde{\Pi}^\pm - 1)\, u^\pm + f^\pm\big) = \mathcal{O}_N(\Gamma_\pm^N)$. Together with Lemma 4.2 and (4.22) this shows that (4.23) holds true. The identity (4.24) follows from (4.19) and (4.23) because $(\tilde{\Pi}^\pm + 1)\, \tilde{\partial}^\pm u^\pm = L^\pm\, u^\pm + \tilde{\partial}^\pm f^\pm - \tilde{\partial}^\pm\big((\tilde{\Pi}^\pm - 1)\, u^\pm + f^\pm\big)$. $\square$

By virtue of Lemma 4.2 we may now define $B_\pm^\nu \in C^\infty(\mathscr{N}_\pm, \mathscr{L}(\mathbb{C}^{d_*}))$, $\nu \in \mathbb{N}_0$, by successively solving the differential equations

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}\big(L^\pm\, B_\pm^\nu + i\tilde{\partial}^\pm(\tilde{\partial}^\pm B_\pm^{\nu-1})\big) \;=\; \mathcal{O}(\Gamma_\pm^N)\,, \qquad B_\pm^{-1} := 0\,, \tag{4.25}$$

on $\mathscr{N}_\pm$ with initial conditions

$$B_\pm^0(0,x,\eta) \;:=\; \chi(x,\eta)\, \Lambda^\pm\big(\mathbf{\Pi}^\pm(0,x,\eta)\big) \;=\; \chi(x,\eta)\, \Lambda^\pm\big(\eta + i\nabla\varphi(x)\big)\,, \tag{4.26}$$

for $(0,x,\eta) \in \mathscr{N}_\pm$, and

$$B_\pm^\nu\big|_{t=0} \;:=\; -(2\Pi_0^\pm)^{-1}\alpha_0\, (i\tilde{\partial}^+ B_+^{\nu-1} + i\tilde{\partial}^- B_-^{\nu-1})\big|_{t=0}\,, \qquad \nu \in \mathbb{N}\,. \tag{4.27}$$

Here χ is the cut-off function introduced below (4.1). We summarize the results of the previous constructions in the following proposition.

Proposition 4.4. *The matrix-valued amplitudes $B_\pm^\nu \in C^\infty(\mathscr{N}_\pm, \mathscr{L}(\mathbb{C}^{d_*}))$ defined by* (4.25)–(4.27) *solve the original transport equations* $(\mathbf{T_0}), (\mathbf{T_1}), (\mathbf{T_2}), \ldots$ *on $\mathscr{N}_\pm$ and*

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)}(\tilde\Pi^\pm + 1)\,\tilde\partial^\pm B_\pm^\nu \,=\, \mathcal{O}(\Gamma_\pm^N)\,, \qquad N \in \mathbb{N}\,,\ \alpha \in \mathbb{N}_0^{3d+1}. \tag{4.28}$$

Moreover,

$$\partial^\alpha_{(t,\Re x,\Im x,\eta)} d_{\overline{x}} B_\pm^\nu \,=\, \mathcal{O}(\Gamma_\pm^N)\,, \qquad N \in \mathbb{N}\,,\ \alpha \in \mathbb{N}_0^{3d+1}. \tag{4.29}$$

The supports of $B_\pm^\nu$, $\nu \in \mathbb{N}_0$, are compact and contained in some fixed compact neighborhood of $\mathscr{D}_\pm$.

Proof. We argue by induction successively applying Lemma 4.3. For $\nu = 0$, we set $f^\pm = 0$ so that (4.18) is satisfied trivially and (4.19) is just (4.25). The initial condition (4.22) is implied by (4.26). By Lemma 4.3 we see that (4.28) is valid, for $\nu = 0$, and that $\partial^\alpha_{(t,\Re x,\Im x,\eta)}(\tilde\Pi^\pm - 1)\,B_\pm^0 = \mathcal{O}(\Gamma_\pm^N)$, which implies $\partial^\alpha_{(t,\Re x,\Im x,\eta)}\Lambda^\mp(\nabla_x\psi_\pm + i\nabla\varphi)\,B_\pm^0 = \mathcal{O}(\Gamma_\pm^N)$, which is $(\mathbf{T_0})$. Next, assume that $n \in \mathbb{N}_0$ and that $B_\pm^\nu$ fulfills $(\mathbf{T}_{\boldsymbol{\nu}})$, (4.28), and (4.29), for every $\nu = 0, \ldots, n$. Then (4.18)–(4.21) are fulfilled with $f^\pm = i\tilde\partial^\pm B_\pm^n$ and $u^\pm = B_\pm^{n+1}$ by (4.28) and the definition of $B_\pm^{n+1}$. At $t = 0$ we have

$$\begin{aligned}(\tilde\Pi^\pm - 1)\,B_\pm^{n+1}\big|_{t=0} &= (-2\Pi_0^\pm)^{-1}\big(\tilde\Pi^\pm - 1\big)\,\alpha_0\,(i\tilde\partial^+ B_+^n + i\tilde\partial^- B_-^n)\big|_{t=0}\\ &= (-2\Pi_0^\pm)^{-1}\big(\Pi_0^\pm + \boldsymbol{\alpha}\cdot\mathbf{\Pi}^\pm - \alpha_0\big)\,(i\tilde\partial^+ B_+^n + i\tilde\partial^- B_-^n)\big|_{t=0}\,.\end{aligned} \tag{4.30}$$

Furthermore, we observe that (4.28) with $\nu = n$ yields

$$\partial^\beta_{(\Re x,\Im x,\eta)}\big\{\Pi_0^\pm\,(i\tilde\partial^\pm B_\pm^n)\big|_{t=0} - (\boldsymbol{\alpha}\cdot\mathbf{\Pi}^\pm - \alpha_0)\,(i\tilde\partial^\pm B_\pm^n)\big|_{t=0}\big\} = \mathcal{O}(|\Im x|^N)\,, \tag{4.31}$$

because $\Gamma(0,x,\eta) = |\Im(x,\eta)|^2 - \langle\,\Im x\,|\,\Re\eta\,\rangle + \Im\langle\,\eta\,|\,x\,\rangle = |\Im x|^2$, for real η, where

$$\mathbf{\Pi}^\pm\big|_{t=0} = \eta + i\nabla\varphi(x) \tag{4.32}$$

is independent of the choice of the $\pm$-signs. Combining (4.30)-(4.32) we arrive at

$$\begin{aligned}&\partial^\beta_{(\Re x,\Im x,\eta)}\big\{(\tilde\Pi^\pm - 1)\,B_\pm^{n+1}\big|_{t=0}\big\}\\ &= \partial^\beta_{(\Re x,\Im x,\eta)}\big\{(-2\Pi_0^\pm)^{-1}\,\big((\Pi_0^\pm + \Pi_0^+)(i\tilde\partial^+ B_+^n) + (\Pi_0^\pm + \Pi_0^-)(i\tilde\partial^- B_-^n)\big)\big|_{t=0}\big\}\\ &= \partial^\beta_{(\Re x,\Im x,\eta)}\big\{-i\tilde\partial^\pm B_\pm^n\big|_{t=0}\big\} \qquad \mathrm{mod}\ \mathcal{O}(|\Im x|^N)\,, \quad \beta \in \mathbb{N}_0^{3d}\,.\end{aligned}$$

In the last step we used

$$\Pi_0^+\big|_{t=0} = \sqrt{1 + (\eta + i\nabla\varphi)^2} = -\Pi_0^-\big|_{t=0}\,. \tag{4.33}$$

In conclusion we see that (4.22) is satisfied, too. Again by Lemma 4.3 we deduce that (4.28) is fulfilled, for $\nu = n+1$, and that $\partial^\alpha_{(t,\Re x,\Im x,\eta)}((\tilde\Pi^\pm - 1)\,B_\pm^{n+1} + i\tilde\partial^\pm B_\pm^n) = \mathcal{O}(\Gamma_\pm^N)$. By virtue of Lemma 4.1 we conclude that $B_\pm^{n+1}$ fulfills the

differential equation in $(\mathbf{T_{n+1}})$. The initial condition in $(\mathbf{T_{n+1}})$ is satisfied because of (4.27) and (4.33).

Finally, we recall that the estimates (4.29) follow from Lemma 4.2. The last assertion on the supports of $B_\pm^\nu$ is clear from the constructions in Lemma 4.2, the fact that the values of t are bounded on $\mathscr{N}_\pm$, and the fact that the initial conditions $B_\pm^\nu|_{t=0}$ are supported in $\mathrm{supp}(\chi)$. $\square$

In order to calculate the leading asymptotics of the Green kernel of $D_{h,V}$ at $(x_\star, y_\star)$ we have to compute the value of $B_+^0(\tau, x, 0)$, for $(\tau, x, 0) \in \mathring{\mathscr{D}}^+$, more explicitly. We recall that, by definition, $(\tau, x, 0) \in \mathring{\mathscr{D}}^+$ implies that there is some $y \in K_0$ such that $Q^+(t, y, 0) \in K_0$, $t \in [0, \tau]$, and $Q^+(\tau, y, 0) = x$.

Lemma 4.5. *Let B_+^0 be a solution of $(\mathbf{T_0})$ as in* (4.25) *and* (4.26). *Let $(\tau, x, 0) \in \mathring{\mathscr{D}}^+$ and let $U(\cdot, y) : [0, \tau] \to \mathscr{L}(\mathbb{C}^{d_*})$ be the solution of*

$$\frac{d}{dt} U(t, y) \;=\; -\frac{i\boldsymbol{\alpha}}{2} \cdot \frac{\nabla V(Q^+(t, y, 0))}{V(Q^+(t, y, 0))}\, U(t, y)\,, \qquad U(0, y) = \mathbb{1}\,,$$

on $[0, \tau]$, where $Q^+(\tau, y, 0) = x$. Then

$$B_+^0(\tau, x, 0) = \frac{(-V(y))^{1/2}\, \chi(y, 0)}{(-V(x))^{1/2} \det\big[d_y Q^+(\tau, y, 0)\big]^{1/2}}\, U(\tau, y)\, \Lambda^+(i\nabla\varphi(y)). \tag{4.34}$$

Proof. On account of Proposition 3.9 we have $\psi_+(t, x, 0) = 0$, $\nabla_x \psi_+(t, x, 0) = 0$, thus $\Pi_0^+(t, x, 0) = \sqrt{1 - \nabla\varphi(x)^2} = -V(x)$ and $\mathbf{\Pi}^+(t, x, 0) = i\nabla\varphi(x)$, for all $(t, x, 0) \in \mathscr{D}^+$. We further deduce that $\partial_t \Pi_0^+(t, x, 0) = 0$, $\mathrm{div}_x \mathbf{\Pi}^+(t, x, 0) = i\Delta\varphi(x)$, $\partial_i^+ \Pi_j^+(t, x, 0) = -i\partial_{x_i}\partial_{x_j}\varphi(x)$, $\partial_0^+ \Pi_j^+(t, x, 0) = 0$, $\partial_j^+ \Pi_0^+(t, x, 0) = \partial_{x_j} V(x)$, where $i, j \in \{1, \dots, d\}$, thus

$$-i \sum_{0 \leqslant \mu < \nu \leqslant d} \sigma_{\mu\nu}\, (\mathrm{rot}^+ \mathbf{\Pi}^+)_{\mu\nu}(t, x, 0) = \gamma_0\, \boldsymbol{\gamma} \cdot (-\nabla V(x)) \;=\; \boldsymbol{\alpha} \cdot \nabla V(x)\,,$$

for all $(t, x, 0) \in \mathscr{D}^+$, where $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_d)$. For $(t, y, 0) \in \widetilde{\mathscr{D}}^+$, the differential equation (4.12) determining B_+^0, for the choice $S^+ = 0$, thus reads

$$\partial_t \widetilde{B}_+^0(t, y, 0) = \frac{i\Delta\varphi(Q^+(t, y, 0)) + \boldsymbol{\alpha} \cdot \nabla V(Q^+(t, y, 0))}{2iV(Q^+(t, y, 0))}\, \widetilde{B}_+^0(t, y, 0)\,, \tag{4.35}$$

for $t \in [0, \tau]$, and we have the initial condition $\widetilde{B}_+^0(0, y, 0) = \chi(y, 0)\, \Lambda^+(i\nabla\varphi(y))$. Using the ansatz $\widetilde{B}_+^0(t, y, 0) = \widetilde{b}(t, y)\, U(t, y)\, \widetilde{B}_+^0(0, y, 0)$ with a scalar $\widetilde{b}$ it thus remains to solve

$$\partial_t \widetilde{b}(t, y) \;=\; \frac{\Delta\varphi(Q^+(t, y, 0))}{2V(Q^+(t, y, 0))}\, \widetilde{b}(t, y)\,, \quad t \in [0, \tau]\,, \quad \widetilde{b}(0, y) = \chi(y, 0)\,.$$

Using Liouville's formula for the equation $\partial_t Q^+(t,y,0) = F(Q^+(t,y,0))$, where $F(x) := \nabla_p H(x, \nabla\varphi(x)) = -V^{-1}(x)\,\nabla\varphi(x)$, $x \in K_0$, that is,

$$\partial_t \det[d_y Q^+(t,y,0)] \;=\; \operatorname{div} F(Q^+(t,y,0))\,\det[d_y Q^+(t,y,0)]\,, \quad (t,y,0) \in \widetilde{\mathscr{D}}^+,$$

where $\mathrm{div} F = -V^{-1}\Delta\varphi - V^{-1}\nabla V \cdot F$, it is, however, elementary to verify that

$$\begin{aligned}
&\partial_t\big\{(-V(Q^+(t,y,0)))^{-1/2}\,\det[d_y Q^+(t,y,0)]^{-1/2}\big\}\\
&\quad= \frac{\Delta\varphi(Q^+(t,y,0))}{2V(Q^+(t,y,0))}\,\big\{(-V(Q^+(t,y,0)))^{-1/2}\,\det[d_y Q^+(t,y,0)]^{-1/2}\big\}\,.
\end{aligned}$$

We deduce that $\widetilde{b}(t,y)$ is equal to the term in the curly brackets $\{\cdots\}$ times $(-V(Q^+(0,y,0))^{1/2} = (-V(y))^{1/2}$ (so that $\widetilde{b}(0,y) = 1$) and the formula (4.34) follows. □

In the one-dimensional case the formula (4.34) for the solution of the first transport equation can still be written a bit more explicitly.

Lemma 4.6. *Assume that $d = 1$, let B^0_+ be a solution of $(\mathbf{T_0})$ as in (4.25) and (4.26), and let $(\tau, x, 0) \in \mathring{\mathscr{D}}^+$. Then*

$$\begin{aligned}
B^0_+(\tau,x,0) \;&= \frac{(-V(y))^{1/2}\,\chi(y,0)}{(-V(x))^{1/2}\,(Q^+)'_y(\tau,y,0)^{1/2}}\\
&\quad\cdot \big(\cos(\vartheta(\tau))\,\mathbb{1} - i\sin(\vartheta(\tau))\,\alpha_1\big)\,\Lambda^+(i\varphi'(y)),
\end{aligned} \tag{4.36}$$

where $Q^+(\tau,y,0) = x$ and

$$\vartheta(t) \;:=\; \int_0^t \frac{V'(Q^+(s,y,0))}{2V(Q^+(s,y,0))}\,ds\,, \qquad t \in [0,\tau]\,.$$

Proof. On account of Lemma 4.5 we just have to compute the solution of

$$\frac{d}{dt}\,U(t,y) \;=\; -i\,\frac{V'(Q^+(t,y,0))}{2V(Q^+(t,y,0))}\,\alpha_1\,U(t,y)\,, \quad t \in [0,\tau]\,, \quad U(0,y) \;=\; \mathbb{1}\,,$$

which is given by $U(t,y) = \cos(\vartheta(t))\,\mathbb{1} - i\sin(\vartheta(t))\,\alpha_1$, $t \in [0,\tau]$. □

Remark 4.7. We know that $\tilde{\Pi}^+(\tau,x_\star,0) \;=\; -\alpha_0\,\boldsymbol{\alpha}\cdot i\nabla\varphi(x_\star) + \alpha_0\,(-V(x_\star))$, where $\nabla\varphi(x_\star) = \varpi(\tau)$. Then the identity $(\tilde{\Pi}^+ - 1)\,B^0_+(\tau,x_\star,0) = 0$ implies

$$\begin{aligned}
B^+_0(\tau,x_\star,0) \;&=\; \Big(\frac{1}{2} + \frac{1}{2}\,\tilde{\Pi}^+(\tau,x_\star,0)\Big)\,B^+_0(\tau,x_\star,0)\\
&=\; (-V(x_\star))\,\Lambda(i\varpi(\tau))\,\alpha_0\,B^+_0(\tau,x_\star,0)\,.
\end{aligned}$$

In the following we verify directly that taking the hermitian conjugate of $M(x_\star,y_\star) \;:=\; (-V(x_\star))\Lambda^+(i\varpi(\tau))\,\alpha_0\,U(\tau)\,(-V(y_\star))\Lambda^+(i\varpi(0))$ gives the same expression as interchanging the roles of $x_\star$ and $y_\star$.

Notice that the Hamiltonian trajectory in $\{H = 0\}$ whose position space projection runs from $x_\star$ to $y_\star$ is given by $\binom{\widetilde{\gamma}}{\widetilde{\varpi}}(t) := \binom{\gamma}{-\varpi}(\tau - t)$, $t \in [0, \tau]$. Thus

$$M(y_\star, x_\star) \;=\; (-V(y_\star))\Lambda^+(-i\varpi(0))\,\alpha_0\,\widetilde{U}(\tau)\,(-V(x_\star))\Lambda^+(-i\varpi(\tau))\,,$$

where $\widetilde{U}$ solves

$$\frac{d}{dt}\,\widetilde{U}(t) \;=\; i\boldsymbol{\alpha}\cdot w(\widetilde{\gamma}(t))\,\widetilde{U}(t)\,,\quad t \in [0,\tau]\,,\qquad w := -(2V)^{-1}\nabla V\,.$$

Hence, in order to verify $M(y_\star, x_\star) = M(x_\star, y_\star)^*$ it suffices to show that $\alpha_0\,\widetilde{U}(\tau) = U(\tau)^*\,\alpha_0$. To this end we recall that $U(\tau)$ is given by the Dyson series $U(\tau) = \sum_{n=0}^\infty I_n(\tau)$, where

$$I_n(\tau) \;:=\; \int_{\tau\,\triangle_n} i\boldsymbol{\alpha}\cdot w(\gamma(t_n))\cdots i\boldsymbol{\alpha}\cdot w(\gamma(t_1))\,dt_1\ldots dt_n\,,$$

$\tau\,\triangle_n := \{0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant \tau\}$ denoting the n-dimensional standard simplex scaled by τ. On the one hand, $\{\alpha_0, \alpha_j\} = 0$ and $\alpha_j^* = \alpha_j$, $j = 1, \ldots, d$, implies

$$I_n(\tau)^*\,\alpha_0 \;=\; \alpha_0 \int_{\tau\,\triangle_n} i\boldsymbol{\alpha}\cdot w(\gamma(t_1))\cdots i\boldsymbol{\alpha}\cdot w(\gamma(t_n))\,dt_1\ldots dt_n\,.$$

On the other hand the substitution $s_1 = \tau - t_n, \ldots, s_n = \tau - t_1$ turns the latter expression into

$$\alpha_0 \int_{\tau\,\triangle_n} i\boldsymbol{\alpha}\cdot w(\widetilde{\gamma}(s_n))\cdots i\boldsymbol{\alpha}\cdot w(\widetilde{\gamma}(s_1))\,ds_1\ldots ds_n =: \alpha_0\,\widetilde{I}_n(\tau)\,,$$

where $\widetilde{U}(\tau) = \sum_{n=0}^\infty \widetilde{I}_n(\tau)$. $\diamond$

4.3. **Approximate solution of $(\pm \boldsymbol{h\partial_t} + \boldsymbol{D_{h,V,\varphi}})\,\boldsymbol{u_\pm = 0}$.** Finally, we put the results of Section 3 and the present section together. To this end we pick smooth cut-off functions, $\varrho_\pm \in C^\infty(\mathbb{R}_0^+ \times \mathbb{R}^{2d})$, such that $\varrho_\pm \equiv 1$ in some neighborhood of $\mathscr{E}_\pm$ and $\mathrm{supp}(\varrho_\pm) \subset \mathscr{N}_\pm \cap (\mathbb{R}_0^+ \times \mathbb{R}^{2d})$ and such that all partial derivatives of any order of $\varrho_\pm$ are uniformly bounded. Furthermore, we define $B_\pm(\,\cdot\,; h) \in C_0^\infty(\mathbb{R}_0^+ \times \mathbb{R}^{2d}, \mathscr{L}(\mathbb{C}^{d_*}))$, $h \in (0,1]$, by

$$B_\pm(t,x,\eta;h) \;=\; \sum_{\nu=0}^\infty h^\nu\,\theta(h/\varepsilon_\nu)\,\varrho_\pm(t,x,\eta)\,B_\pm^\nu(t,x,\eta)\,, \tag{4.37}$$

for $(t,x,\eta) \in \mathbb{R}_0^+ \times \mathbb{R}^{2d}$, $h \in (0,1]$, where $\theta \in C^\infty(\mathbb{R},[0,1])$ equals 1 on $(-\infty, 1]$ and 0 on $[2,\infty)$. ($B_\pm^\nu$ is set equal to zero outside its original domain of definition $\mathscr{N}_\pm$.) If $\varepsilon_\nu \searrow 0$ sufficiently fast, we have

(4.38)

$$\sup_{\substack{(t,x,\eta)\in\\ \mathbb{R}_0^+\times\mathbb{R}^{2d}}} \Big\|\partial^\alpha_{(t,x,\eta)}\Big(B_\pm(t,x,\eta;h) - \sum_{\nu=0}^N h^\nu\,\varrho_\pm(t,x,\eta)\,B_\pm^\nu(t,x,\eta)\Big)\Big\| \leqslant C_{N,\alpha}\,h^{N+1},$$

for $N \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^{2d+1}$, $h \in (0, h_{\alpha,N}]$, and suitable constants $C_{N,\alpha}, h_{\alpha,N} \in (0,\infty)$.

Proposition 4.8. *There exist compactly supported matrix-valued functions, $\check{r}_\pm(\,\cdot\,;h) \in C^\infty(\mathbb{R}_0^+ \times \mathbb{R}^{2d}, \mathscr{L}(\mathbb{C}^{d_*}))$, such that*

$$(\pm h\,\partial_t + D_{h,V,\varphi})\,(e^{i\psi_\pm/h}\,B_\pm) = \check{r}_\pm\,, \qquad h \in (0,1]\,,$$

and, for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^{2d+1}$, there is some $C_{N,\alpha} \in (0,\infty)$ such that

$$\sup_{(t,x,\eta)\in\mathbb{R}_0^+\times\mathbb{R}^{2d}} \|\partial^\alpha_{(t,x,\eta)}\check{r}_\pm(t,x,\eta;h)\| \leqslant C_{N,\alpha}\,h^N, \quad h \in (0,1]\,. \tag{4.39}$$

Proof. Since all terms in (4.1) corresponding to the different powers of h are equal to some smooth matrix-valued function on $\mathscr{N}_\pm$ whose partial derivatives of any order are equal to $\mathcal{O}(\Gamma_\pm^N)$, $N \in \mathbb{N}$, and since $\Gamma_\pm \leqslant \mathcal{O}(1)\,\Im\psi_\pm$ on the real domain,

$$\begin{aligned}(\pm h\,\partial_t + D_{h,V,\varphi})\,[e^{i\psi_\pm/h}\,B_\pm] &= \sum_{\nu=0}^\infty h^\nu\;\theta(h/\varepsilon_\nu)\,e^{i\psi_\pm/h}\,\mathcal{O}\big((\Im\psi_\pm)^N\big)\\ &\quad+\sum_{\nu=1}^\infty h^\nu\,\theta(h/\varepsilon_\nu)\,e^{i\psi_\pm/h}\,[(\pm\partial_t - i\boldsymbol{\alpha}\cdot\nabla)\varrho_\pm]\,B_\pm^{\nu-1}\\ &\quad+\sum_{\nu=1}^\infty h^\nu\,\big(\theta(h/\varepsilon_{\nu-1}) - \theta(h/\varepsilon_\nu)\big)\,e^{i\psi_\pm/h}\,(\pm\partial_t - i\boldsymbol{\alpha}\cdot\nabla)[\varrho_\pm\,B_\pm^{\nu-1}]\,.\end{aligned} \tag{4.40}$$

Consequently, all partial derivatives of the first term on the right hand side of (4.40) are of order $\mathcal{O}(h^\infty)$ because

$$e^{-\Im\psi_\pm/h}\,(\Im\psi_\pm)^N \leqslant N!\,h^N, \qquad N \in \mathbb{N}\,. \tag{4.41}$$

Furthermore, $\Im\psi_\pm > 0$ on $\mathrm{supp}(\varrho'_\pm)$ and all partial derivatives of $R_\nu := [(\pm\partial_t - i\boldsymbol{\alpha}\cdot\nabla)\varrho_\pm]\,B_\pm^\nu$ are locally bounded, whence $\partial^\alpha_{(t,x,\eta)}[e^{i\psi_\pm/h}\,R_\nu] = \mathcal{O}(h^\infty)$. All partial derivatives of the third term on the right hand side of (4.40) are of order $\mathcal{O}(h^\infty)$, too, since $\theta(h/\varepsilon_{\nu-1}) - \theta(h/\varepsilon_\nu) = 0$, for all $h \in (0,\varepsilon_\nu)$, $\nu \in \mathbb{N}$. ☐

5. A parametrix for $D_{h,V,\varphi}$

Given $k, m \in \mathbb{R}$, we write $b \in S^k(\langle\xi\rangle^m; \mathscr{L}(\mathbb{C}^{d_*}))$, for some map $b : \mathbb{R}^{2d}\times(0,h_0] \to \mathscr{L}(\mathbb{C}^{d_*})$, if $h_0 > 0$, $b(\,\cdot\,;h) \in C^\infty(\mathbb{R}^{2d}, \mathscr{L}(\mathbb{C}^{d_*}))$, $h \in (0,h_0]$, and, for all $\alpha \in \mathbb{N}_0^{2d}$, we find $h_\alpha, C_\alpha \in (0,\infty)$ such that

$$\|\partial^\alpha_{(x,\xi)}b(x,\xi;h)\| \leqslant C_\alpha\,\langle\xi\rangle^m\,h^{-k}, \qquad x,\xi \in \mathbb{R}^d,\ h \in (0,h_\alpha]\,.$$

We further set

$$S^{-\infty}(\langle\xi\rangle^{-\infty}; \mathscr{L}(\mathbb{C}^{d_*})) := \bigcap_{k\in\mathbb{N}} S^{-k}(\langle\xi\rangle^{-k}; \mathscr{L}(\mathbb{C}^{d_*}))\,.$$

In what follows we work with the semi-classical standard quantization of matrix-valued symbols $b \in S^k(\langle\xi\rangle^m; \mathscr{L}(\mathbb{C}^{d_*}))$ determined by the oscillatory integrals

$$\mathrm{Op}_h(b)\, f(x) := \int e^{i\langle\,\xi\,|\,x-y\,\rangle/h}\, b(x,\xi)\, f(y) \frac{dyd\xi}{(2\pi h)^d}\,, \qquad f \in \mathscr{S}(\mathbb{R}^d, \mathbb{C}^{d_*})\,.$$

Let $k_1, k_2, m_1, m_2 \in \mathbb{R}$. We recall that, for $b \in S^{k_1}(\langle\xi\rangle^{m_1}; \mathscr{L}(\mathbb{C}^{d_*}))$ and $c \in S^{k_2}(\langle\xi\rangle^{m_2}; \mathscr{L}(\mathbb{C}^{d_*}))$, the symbol, $b\#_h c \in S^{k_1+k_2}(\langle\xi\rangle^{m_1+m_2}; \mathscr{L}(\mathbb{C}^{d_*}))$, of $\mathrm{Op}_h(b) \circ \mathrm{Op}_h(c)$ has the following asymptotic expansion in $S^{k_1+k_2}(\langle\xi\rangle^{m_1+m_2}; \mathscr{L}(\mathbb{C}^{d_*}))$,

$$b\#_h c(x,\xi;h) = e^{ihD_\eta D_y}\, b(x,\eta)\, c(y,\xi)\big|_{\substack{y=x\\ \eta=\xi}} \asymp \sum_{\alpha\in\mathbb{N}_0^d} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!}\, (\partial_\xi^\alpha b)(x,\xi)\, (\partial_x^\alpha c)(x,\xi)\,.$$

If the symbol $\widehat{D}_{V,\varphi}(x,\xi) \in S^0(\langle\xi\rangle; \mathscr{L}(\mathbb{C}^{d_*}))$ defined in (2.15) were invertible, for every $(x,\xi) \in \mathbb{R}^{2d}$, we had a well-known asymptotic expansion, $\breve{q}(x,\xi) \asymp \sum_{\nu=0}^\infty h^\nu\, q_\nu(x,\xi)$, of the matrix-valued symbol of the inverse operator. We can, however, write down this asymptotic expansion formally and determine $q_\nu(x,\xi)$ at every point $(x,\xi) \in \mathbb{R}^{2d}$ where $\widehat{D}_{V,\varphi}(x,\xi)$ is invertible. We proceed in this way and pick some cut-off function $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^{2d}, [0,1])$ such that $\widetilde{\chi} \equiv 1$ in a small neighborhood of $K_0 \times \{0\}$ and $\mathrm{supp}(\widetilde{\chi}) \subset \{\chi = 1\}$, where χ has been introduced below (4.1). Then it turns out that $q_\nu\,(1-\widetilde{\chi}) \in S^0(\langle\xi\rangle^{-\nu-1}; \mathscr{L}(\mathbb{C}^{d_*}))$. Let $\tilde{q}$ be a Borel resummation of $\sum_{\nu=0}^\infty h^\nu\, q_\nu\,(1-\widetilde{\chi})$, so that

$$\tilde{q} - \sum_{\nu=0}^{N-1} h^\nu\, q_\nu\,(1-\widetilde{\chi}) \in S^{-N}(\langle\xi\rangle^{-N-1}; \mathscr{L}(\mathbb{C}^{d_*}))\,, \qquad N \in \mathbb{N}\,.$$

Then $D_{h,V,\varphi} \circ \mathrm{Op}_h(\tilde{q})$ is a pseudo-differential operator whose symbol has the asymptotic expansion

$$\widehat{D}_{V,\varphi}\#_h\tilde{q}(x,\xi) \asymp 1 - \widetilde{\chi}(x,\xi) + \sum_{\nu=0}^\infty h^\nu\, \breve{r}_\nu(x,\xi)\,.$$

Here each error term $\breve{r}_\nu$ contains some partial derivative of $\widetilde{\chi}$ whence $\mathrm{supp}(\breve{r}_\nu) \subset \{\chi = 1\}$, for every $\nu \in \mathbb{N}_0$. Setting

$$q := \tilde{q}\#_h(1-\chi)$$

we thus have

$$\widehat{D}_{V,\varphi}\#_h q - (1-\chi) \in S^{-\infty}(\langle\xi\rangle^{-\infty}; \mathscr{L}(\mathbb{C}^{d_*}))\,. \tag{5.1}$$

Next, we define an operator $\mathcal{P}_h : \mathscr{S}(\mathbb{R}^d, \mathbb{C}^{d_*}) \to \mathscr{S}'(\mathbb{R}^d, \mathbb{C}^{d_*})$,

$$\begin{aligned}(\mathcal{P}_h\, f)(x) := \sum_{\sharp\in\{+,-\}} \sharp \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} e^{i\psi_\sharp(t,x,\eta)/h - i\langle\,\eta\,|\,y\,\rangle/h}\, B_\sharp(t,x,\eta;h)\, f(y)\, \frac{dyd\eta dt}{(2\pi h)^d h} \\ + \mathrm{Op}_h(q)\, f(x)\,, \qquad x \in \mathbb{R}^d,\ f \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^{d_*})\,.\end{aligned} \tag{5.2}$$

Here the integrals in the first line are effectively evaluated over some compact set so that $\mathcal{P}_h$ is obviously well-defined. Furthermore, it is clear that $\mathcal{P}_h$ can be represented as an integral operator with kernel

$$\begin{aligned}\mathcal{P}_h(x,y) = \sum_{\sharp\in\{+,-\}} \sharp \int_0^\infty\int_{\mathbb{R}^d} e^{i\psi_\sharp(t,x,\eta)/h - i\langle\,\eta\,|\,y\,\rangle/h}\, B_\sharp(t,x,\eta;h)\,\frac{d\eta dt}{(2\pi h)^d h} \\ + \check{q}(x,x-y)\,. \qquad (5.3)\end{aligned}$$

We recall that $\check{q}(x,y-x)$, with $\check{q}(x,y) = (\mathcal{F}_h^{-1})_{\xi\to y} q(x,y)$, is the distribution kernel of $\mathrm{Op}_h(q)$. Here we normalize the (component-wise) semi-classical Fourier transform as

$$\hat{f}(\eta) := (\mathcal{F}_h\, f)(\eta) := \int_{\mathbb{R}^d} e^{-i\langle\,\eta\,|\,y\,\rangle/h}\, f(y)\,dy\,, \qquad \eta\in\mathbb{R}^d,\ f\in L^1(\mathbb{R}^d,\mathscr{V})\,,$$

where $\mathscr{V}$ is $\mathbb{C}^{d_*}$ or $\mathscr{L}(\mathbb{C}^{d_*})$. Integrating by parts by means of the operators

$$\frac{1 - ih\,(\overline{\nabla_\eta\psi_\pm} - y)\cdot\nabla_\eta}{1 + |\nabla_\eta\psi_\pm - y|^2}$$

and using the fact that $B_\pm$ is compactly supported it is easy to see that

$$\|\partial_x^\alpha\partial_y^\beta\mathcal{P}_h(x,y)\| \leqslant C_{N,\alpha,\beta}\, h^{-|\alpha|-|\beta|}\,\langle x\rangle^{-N}\,\langle y\rangle^{-N}, \qquad x,y\in\mathbb{R}^d,$$

for all $N\in\mathbb{N}$, $\alpha,\beta\in\mathbb{N}_0^d$, and suitable constants $C_{N,\alpha,\beta}\in(0,\infty)$.

Theorem 5.1. *(i) There is some $\tilde{r}\in S^{-\infty}(\langle\xi\rangle^{-\infty};\mathscr{L}(\mathbb{C}^{d_*}))$, such that*

$$D_{h,V,\varphi}\,\mathcal{P}_h \;=\; \mathbb{1} - \mathrm{Op}_h(\tilde{r})\,.$$

(ii) There exist $h_0\in(0,1]$ and smooth kernels, $\mathcal{R}_h\in C^\infty(\mathbb{R}^{2d},\mathscr{L}(\mathbb{C}^{d_*}))$, $h\in(0,h_0]$, such that

$$\|\partial_x^\alpha\partial_y^\beta\mathcal{R}_h(x,y)\| \leqslant C_{N,\alpha,\beta}\, h^N\langle x\rangle^{-N}\langle y\rangle^{-N}, \qquad x,y\in\mathbb{R}^d,\ h\in(0,h_0]\,, \tag{5.4}$$

for all $N\in\mathbb{N}$, $\alpha,\beta\in\mathbb{N}_0^d$, and suitable constants $C_{N,\alpha,\beta}\in(0,\infty)$, and such that

$$D_{h,V}^{-1}(x,y) \;=\; e^{-(\varphi(x)-\varphi(y))/h}\,(\mathcal{P}_h(x,y) + \mathcal{R}_h(x,y))\,, \qquad x\neq y\,.$$

Proof. (i): By Proposition 4.8 we have, for $f \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^{d_*})$,

$$
\begin{aligned}
D_{h,V,\varphi} &\sum_{\sharp\in\{+,-\}} \sharp \int_0^\infty \int_{\mathbb{R}^d} e^{i\psi_\sharp(t,x,\eta)/h}\, B_\sharp(t,x,\eta;h)\, \hat{f}(\eta)\, \frac{d\eta\, dt}{(2\pi h)^d\, h} \\
&= - \sum_{\sharp\in\{+,-\}} \int_{\mathbb{R}^d} \int_0^\infty \partial_t\big(e^{i\psi_\sharp(t,x,\eta)/h}\, B_\sharp(t,x,\eta;h)\big)\, \hat{f}(\eta)\, \frac{dt\, d\eta}{(2\pi h)^d} \\
&\quad + \sum_{\sharp\in\{+,-\}} \sharp \int_0^\infty \int_{\mathbb{R}^d} \check{r}_\sharp(t,x,\eta;h)\, \hat{f}(\eta)\, \frac{d\eta\, dt}{(2\pi h)^d\, h} \\
&= \int_{\mathbb{R}^d} \chi(x,\eta)\, e^{i\langle\,\eta\,|\,x\,\rangle/h} \sum_{\sharp\in\{+,-\}} \Lambda^\sharp\big(\eta + i\nabla\varphi(x)\big)\, \hat{f}(\eta)\, \frac{d\eta}{(2\pi h)^d} \\
&\quad - \mathrm{Op}_h(\tilde{r}_1) f(x)\,, \qquad (5.5)
\end{aligned}
$$

where $\Lambda^+ + \Lambda^- = \mathbb{1}$. We recall that the integral appearing here in the first line is effectively an integral over some compact set. In particular, the boundary at ∞ does not give any contribution to the integral $\int_0^\infty \partial_t(\cdots)\, dt$. Moreover, we put

$$
\tilde{r}_1(x,\eta;h) := - \sum_{\sharp\in\{+,-\}} \sharp \int_0^\infty e^{-i\langle\,\eta\,|\,x\,\rangle/h}\, \check{r}_\sharp(t,x,\eta;h)\, \frac{dt}{h}\,.
$$

In view of (4.39) and since $\check{r}$ has a compact support it is clear that $\tilde{r}_1$ is a symbol in $S^{-\infty}(\langle\eta\rangle^{-\infty}; \mathscr{L}(\mathbb{C}^{d_*}))$. In fact, if some derivative $\partial^\alpha_{(x,\eta)}$ is applied to $\tilde{r}_1$ the inverse powers of h obtained by differentiating the phase $e^{-i\langle\,\eta\,|\,x\,\rangle/h}$ are compensated for by derivatives of $\check{r}_\sharp$, which are of order $\mathcal{O}(h^\infty)$. On account of (5.1) we further have

$$
D_{h,V,\varphi}\, \mathrm{Op}_h(q)\, f(x) \;=\; \int_{\mathbb{R}^d} e^{i\langle\,\eta\,|\,x\,\rangle/h}\, (1 - \chi(x,\eta))\, \hat{f}(\eta)\, \frac{d\eta}{(2\pi h)^d} - \mathrm{Op}_h(\tilde{r}_2) f(x)\,,
$$

for some $\tilde{r}_2 \in S^{-\infty}(\langle\eta\rangle^{-\infty}, \mathscr{L}(\mathbb{C}^{d_*}))$, which proves (i) with $\tilde{r} := \tilde{r}_1 + \tilde{r}_2$.

(ii): For sufficiently small $h > 0$, $\|\mathrm{Op}_h(\tilde{r})\|_{\mathscr{L}(L^2)} \leqslant 1/2$, and $(\mathbb{1} - \mathrm{Op}_h(\tilde{r}))^{-1} = \mathrm{Op}_h(c)$, for some $c \in S^0(1; \mathscr{L}(\mathbb{C}^{d_*}))$. Now, $\mathrm{Op}_h(c)\,(\mathbb{1} - \mathrm{Op}_h(\tilde{r})) = \mathbb{1}$ is equivalent to $\mathrm{Op}_h(c - 1) = \mathrm{Op}_h(c)\,\mathrm{Op}_h(\tilde{r})$ which implies $c - 1 \in S^{-\infty}(\langle\eta\rangle^{-\infty}; \mathscr{L}(\mathbb{C}^{d_*}))$, because $\tilde{r} \in S^{-\infty}(\langle\eta\rangle^{-\infty}; \mathscr{L}(\mathbb{C}^{d_*}))$. Of course, $\mathcal{P}_h \mathrm{Op}_h(c) = \mathcal{P}_h + \mathcal{P}_h \mathrm{Op}_h(c - 1)$ and the distribution kernel of $\mathrm{Op}_h(c - 1)$ – let us call it $\mathcal{K}_h$ – fulfills $\|\partial_x^\alpha \partial_y^\beta \mathcal{K}_h(x,y)\| \leqslant C_{N,\alpha,\beta}\, h^N\, \langle x - y\rangle^{-N}$. Therefore, using the remarks preceding the statement of this theorem it is easy to check that $\mathcal{R}_h := \mathcal{P}_h\, \mathrm{Op}_h(c - 1)$ is an integral operator with smooth kernel – again denoted by the same symbol – such that (5.4) holds true. Using $D_{h,V,\varphi}\,(\mathcal{P}_h + \mathcal{R}_h) = \mathbb{1}$, and $D_{h,V,\varphi} = e^{\varphi/h}\, D_{h,V}\, e^{-\varphi/h}$, where φ is bounded with bounded partial derivatives of any order, we conclude that $e^{-\varphi/h}(\mathcal{P}_h + \mathcal{R}_h)\, e^{\varphi/h} = D_{h,V}^{-1}$. $\square$

6. Calculation of the leading asymptotics

In this section we calculate the asymptotics of the integral kernel (5.3) of the operator $\mathcal{P}_h$ defined in (5.2) at the distinguished points $x_\star$ and $y_\star$ fulfilling Hypothesis 1.2. On account of Theorem 5.1 this will complete the proofs of our main Theorems 1.3 and 1.5. As in the statement of these theorems we let $\binom{\gamma}{\varpi} : [0,\tau] \to \mathbb{R}^{2d}$ denote a smooth curve solving (1.10) and satisfying (1.11) such that $\gamma(0) = y_\star$ and $\gamma(\tau) = x_\star$ and set

$$v_{y_\star} := \frac{d}{dt}\gamma(0)\,, \qquad v_{x_\star} := \frac{d}{dt}\gamma(\tau)\,.$$

Proposition 6.1. *Let $d \in \mathbb{N}$. As $h \in (0,1]$ tends to zero,*

$$\begin{aligned} &D^{-1}_{h,V,\varphi}(x_\star, y_\star) \\ &= \frac{1}{h^d}\Big(\frac{h}{2\pi}\Big)^{\frac{d-1}{2}} \frac{(1+\mathcal{O}(h))\,(-V(y_\star))^{1/2}\,U(\tau,y_\star)\,\Lambda^+(i\nabla\varphi(y_\star))}{(-V(x_\star))^{1/2}\sqrt{\det\begin{pmatrix} 0 & -v_{y_\star}^\top \\ v_{x_\star} & id_\eta Q^+(\tau,y_\star,0)\end{pmatrix}}}\,, \end{aligned} \tag{6.1}$$

where we use the same notation as in (4.34).

Proof. By Theorem 5.1(ii) it suffices to consider only the kernel $\mathcal{P}_h$. First, a standard argument shows that the distribution kernel $\check{q}(x, x-y)$ of $\mathrm{Op}_h(q)$ in (5.3) does not contribute to the asymptotic expansion in (6.1). In fact, $(x-y)^{2N}\check{q}(x,x-y)$ is the inverse Fourier transform of $h^{2N}\,\Delta_\xi^N q(x,\xi)$ at $x-y$, where $\Delta_\xi^N q(x,\xi)$ is absolutely integrable with respect to ξ, for large $N \in \mathbb{N}$.

Next, we consider the integral

$$I_-(x_\star,y_\star) := \int_0^\infty\!\!\int_{\mathbb{R}^d} e^{i\psi_-(t,x_\star,\eta)/h - i\langle\,\eta\,|\,y_\star\,\rangle/h}\,B_-(t,x_\star,\eta;h)\,\frac{d\eta\,dt}{(2\pi h)^d\,h}\,.$$

At $t=0$ we have $\psi_-(0,x_\star,\eta) - \langle\,\eta\,|\,y_\star\,\rangle = \langle\,\eta\,|\,x_\star - y_\star\,\rangle$. Since $x_\star \neq y_\star$ we can thus show by integration by parts with respect to η that

$$\int_0^\varepsilon\!\!\int_{\mathbb{R}^d} e^{i\psi_-(t,x_\star,\eta)/h - i\langle\,\eta\,|\,y_\star\,\rangle/h}\,B_-(t,x_\star,\eta;h)\,\frac{d\eta\,dt}{(2\pi h)^d\,h} = \mathcal{O}(h^\infty)\,,$$

provided that $\varepsilon > 0$ is sufficiently small. Since $\Im\psi_-(t,x_\star,\eta) > 0$, for $t>0$, by (3.42), it further follows that $\int_\varepsilon^\infty\!\int_{\mathbb{R}^d} e^{i\psi_-/h - i\langle\,\eta\,|\,y_\star\,\rangle/h}\,B_-\,d\eta dt = \mathcal{O}(h^\infty)$, for every fixed $\varepsilon > 0$. Consequently, $I_-(x_\star,y_\star) = \mathcal{O}(h^\infty)$.

Finally, we treat the integral

$$I_+(x_\star,y_\star) := \int_0^\infty\!\!\int_{\mathbb{R}^d} e^{i\psi_+(t,x_\star,\eta)/h - i\langle\,\eta\,|\,y_\star\,\rangle/h}\,B_+(t,x_\star,\eta;h)\,\frac{d\eta\,dt}{(2\pi h)^d\,h}\,,$$

which is the only term contributing to the asymptotic expansion. We shall apply a complex stationary phase expansion with respect to the $d+1$ variables

(t, η). The critical points of the phase are given by

$$0 = \partial_t \psi_+(t, x_\star, \eta), \tag{6.2}$$

$$0 = \nabla_\eta \psi_+(t, x_\star, \eta) - y_\star . \tag{6.3}$$

To find the asymptotics of $I_+(x_\star, y_\star)$ it certainly suffices to determine all critical points (t, η) with $\Im\psi_+(t, x_\star, \eta) = 0$. We know, however, that $\Im\psi_+(t, x_\star, \eta) = 0$ implies $t = 0$ or $(t, x_\star, \eta) \in \mathscr{D}^+$. As above, we infer by integration by parts that $\int_0^\varepsilon \int_{\mathbb{R}^d} e^{i\psi_+/h - i\langle \eta \,|\, y_\star \rangle/h} B_+ \, d\eta dt = \mathcal{O}(h^\infty)$, for some sufficiently small $\varepsilon > 0$. The only critical point (t, η) such that $(t, x_\star, \eta) \in \mathscr{D}^+$ is, however, given by $(t, \eta) = (\tau, 0)$. The method of complex stationary phase [6] thus implies that

$$I_+(x_\star, y_\star) = \frac{(2\pi h)^{\frac{d+1}{2}}}{(2\pi h)^d \, h} \frac{e^{i\psi_+(\tau, x_\star, 0)/h} B_+^0(\tau, x_\star, 0) \big(1 + \mathcal{O}(h)\big)}{\sqrt{\det \frac{1}{i} \begin{pmatrix} \partial_t^2 \psi_+ & \partial_t d_\eta \psi_+ \\ \partial_t \nabla_\eta \psi_+ & d_\eta \nabla_\eta \psi_+ \end{pmatrix} (\tau, x_\star, 0)}} . \tag{6.4}$$

Thanks to (3.45) we know that $\psi_+(\tau, x_\star, 0) = 0 = \partial_t^2 \psi_+(\tau, x_\star, 0)$ and differentiating the identity $Q^+(t, k^+(t, x_\star, \eta), \eta) = x_\star$ we obtain

$$d_y Q^+ \, \partial_t k^+(t, x_\star, \eta) + d_{\overline{y}} Q^+ \, \overline{\partial_t k^+(t, x_\star, \eta)} = -\partial_t Q^+ ,$$
$$d_y Q^+ \, d_\eta k^+(t, x_\star, \eta) + d_{\overline{y}} Q^+ \, \overline{d_\eta k^+(t, x_\star, \eta)} = -d_\eta Q^+ ,$$

where all derivatives of Q^+ are evaluated at $(t, k^+(t, x^\star, \eta), \eta)$. Inserting $(t, \eta) = (\tau, 0)$ we infer that

$$d_y Q^+(\tau, y_\star, 0) \, v_{y_\star} = v_{x_\star} , \tag{6.5}$$

$$d_y Q^+(\tau, y_\star, 0) \, \tfrac{1}{i} d_\eta \nabla_\eta \psi_+(\tau, x_\star, 0) = i \, d_\eta Q^+(\tau, y_\star, 0) , \tag{6.6}$$

The determinant appearing in (6.4) is thus equal to

$$\det \begin{pmatrix} 0 & -v_{y_\star}^\top \\ v_{y_\star} & d_y Q^+(\tau, y_\star, 0)^{-1} \, i \, d_\eta Q^+(\tau, y_\star, 0) \end{pmatrix} . \tag{6.7}$$

Multiplying the determinant (6.7) and the one appearing in (4.34), which can be written as

$$\det \big[d_y Q^+(\tau, y_\star, 0) \big] = \det \begin{pmatrix} 1 & 0 \\ 0 & d_y Q^+(\tau, y_\star, 0) \end{pmatrix} ,$$

and using $\chi(y_\star, 0) = 1$, (6.5), and (6.6) we arrive at (6.1). $\square$

Next, we re-write the formula (6.1) in the one-dimensional case.

Proposition 6.2. *For $d = 1$, we have, as $h \searrow 0$,*

$$D_{h,V,\varphi}^{-1}(x_\star, y_\star) = \frac{1 + \mathcal{O}(h)}{h \, (1 - V^2(x_\star))^{1/4} (1 - V^2(y_\star))^{1/4}}$$
$$\cdot \big(\cos(\vartheta(\tau)) \, \mathbb{1} - i \sin(\vartheta(\tau)) \, \alpha_1 \big) \, (-V(y_\star))^{1/2} \, \Lambda(i \nabla \varphi(y_\star)) , \tag{6.8}$$

Proof. In the case $d = 1$ the formula (6.1) reduces to

$$I(x_\star, y_\star) \,=\, (1+\mathcal{O}(h))\,\frac{U(\tau,y_\star)\,(-V(y_\star))^{1/2}\,\Lambda^+(i\varphi'(y_\star))}{h\,|V(x_\star)|^{1/2}|V(y_\star)|^{1/2}(v_{x_\star}\,v_{y_\star})^{1/2}}\,.$$

Here $v_{x_\star}$ and $v_{y_\star}$ have the same sign and $|v_z| = (1-V^2(z))^{1/2}/|V(z)|$, for $z = x_\star, y_\star$, so that the factors $|V(z)|$, $z = x_\star, y_\star$, cancel each other in the denominator. Moreover, we have already calculated $U(\tau, y_\star)$ in the proof of Lemma 4.6. $\square$

In more than one dimension we can evaluate the determinant in (6.1) more explicitly. To this end we first prove the following lemma.

Lemma 6.3. *Let X be the position space projection of the Hamiltonian flow associated with H as defined in* (2.11)*. Then the following identity holds,*

$$id_\eta Q^+(\tau,y_\star,0) \,=\, d_pX(\tau,y_\star,\varpi(0))\,, \tag{6.9}$$

where $\varpi(0) = \nabla\varphi(y_\star)$ is the initial momentum of the Hamiltonian trajectory from $y_\star$ to $x_\star$ as in the statement of Theorem 1.3.

Proof. By Lemma 3.1 we have $Q^+(t,y_\star,0) = X(t,y_\star,0) = \gamma(t)$, $t\in[0,\tau]$. We set $\rho(t) := (\gamma(t),\varpi(t)) = (\gamma(t),\nabla\varphi(\gamma(t)))$, $t \in [0,\tau]$. On account of (2.21), (2.22), and (3.12) we thus find

$$\frac{d}{dt}\begin{pmatrix} d_\eta Q^+\\ d_\eta\Xi^+\end{pmatrix}(t,y_\star,0) = \begin{pmatrix}\mathbb{B}(t) & -i\mathbb{A}(t)\\ 0 & -\mathbb{B}(t)^\top\end{pmatrix}\begin{pmatrix} d_\eta Q^+\\ d_\eta\Xi^+\end{pmatrix}(t,y_\star,0)\,,$$

$$\begin{pmatrix} d_\eta Q^+\\ d_\eta\Xi^+\end{pmatrix}(0,y_\star,0) = \begin{pmatrix}0\\ \mathbb{1}\end{pmatrix},$$

where $\mathbb{A}(t) = H''_{pp}(\rho(t))$ and $\mathbb{B}(t) := \mathbb{B}(t,y_\star)$ is defined as in (3.24) with $y = y_\star$, for $t\in[0,\tau]$. On the other hand,

$$\frac{d}{dt}\begin{pmatrix} d_pX\\ d_pP\end{pmatrix}(t,\rho(0)) = \begin{pmatrix} H''_{px} & H''_{pp}\\ -H''_{xx} & -H''_{xp}\end{pmatrix}(\rho(t))\begin{pmatrix} d_pX\\ d_pP\end{pmatrix}(t,\rho(0)), \tag{6.10}$$

$$\begin{pmatrix} d_pX\\ d_pP\end{pmatrix}(0,\rho(0)) = \begin{pmatrix}0\\ \mathbb{1}\end{pmatrix}. \tag{6.11}$$

Hence, using $0 = d_x(\nabla_x H(x,\nabla\varphi)) = H''_{xx}(x,\nabla\varphi) + H''_{xp}(x,\nabla\varphi)\,\varphi''$ on K_0 and (6.10)&(6.11), we can verify directly that

$$\begin{pmatrix}\widetilde{X}(t)\\ \widetilde{P}(t)\end{pmatrix} := \begin{pmatrix} d_pX(t,\rho(t))\\ -\varphi''(\gamma(t))\,d_pX(t,\rho(t)) + d_pP(t,\rho(t))\end{pmatrix}, \qquad t\in[0,\tau]\,,$$

solves

$$\frac{d}{dt}\begin{pmatrix}\widetilde{X}\\ \widetilde{P}\end{pmatrix}(t) = \begin{pmatrix}\mathbb{B}(t) & \mathbb{A}(t)\\ 0 & -\mathbb{B}(t)^\top\end{pmatrix}\begin{pmatrix}\widetilde{X}\\ \widetilde{P}\end{pmatrix}(t), \quad \begin{pmatrix}\widetilde{X}\\ \widetilde{P}\end{pmatrix}(0) = \begin{pmatrix}0\\ \mathbb{1}\end{pmatrix},$$

which implies (6.9). $\square$

Lemma 6.4. *The following identity holds,*

$$\det\begin{pmatrix} 0 & -v_{y_\star}^\top \\ v_{x_\star} & id_\eta Q^+(\tau, y_\star, 0)\end{pmatrix} = \frac{d_{\mathrm{A}}(x_\star, y_\star)^{d-1}\det\left(\exp'_{y_\star}(\exp_{y_\star}^{-1}(x_\star))\right)}{|V(x_\star)||V(y_\star)|(1-V^2(x_\star))^{\frac{d-2}{2}}(1-V^2(y_\star))^{\frac{d-2}{2}}}.$$

Proof. We drop the subscripts $\star$ of the distinguished points $x_\star$, $y_\star$ in this proof. First, we introduce some notation. Let $b_1, \dots, b_d$ denote some $G(y)$-orthonormal basis of $\mathbb{R}^d$ such that $b_d = (1-V^2(y))^{-1/2} v_y/|v_y|$ and $c_1, \dots, c_d$ be some $G(x)$-orthonormal basis of $\mathbb{R}^d$ such that $c_d = (1-V^2(x))^{-1/2} v_x/|v_x|$. Let $b_1^*, \dots, b_d^*$ and $c_1^*, \dots, c_d^*$ denote the corresponding dual bases and let B and C denote the matrices whose i-th row is b_i^* and c_i^*, respectively. Then we have $B\, v_y = (1-V^2(y))^{1/2}\,|v_y|\,\mathsf{e}_d$ and $C\, v_x = (1-V^2(x))^{1/2}\,|v_x|\,\mathsf{e}_d$, where e_d is the d-th canonical basis vector of $\mathbb{R}^d$. Therefore, we find, using (6.9), that is, $id_\eta Q^+(\tau, y, 0) = d_p X(\tau, y, \varpi(0))$,

$$\begin{aligned}&\det\begin{pmatrix} 1 & 0 \\ 0 & C\end{pmatrix}\det\begin{pmatrix} 0 & -v_y^\top \\ v_x & id_\eta Q^+(\tau, y, 0)\end{pmatrix}\det\begin{pmatrix} 1 & 0 \\ 0 & B^\top\end{pmatrix} \\ &= (1-V^2(x))^{1/2}\,|v_x|\,(1-V^2(y))^{1/2}\,|v_y|\,\det\begin{pmatrix} 0 & -\mathsf{e}_d^\top \\ \mathsf{e}_d & C\, d_p X(\tau, y, \varpi(0))\, B^\top\end{pmatrix}.\end{aligned}$$

Since $\det B = (1-V^2(y))^{d/2}$, $\det C = (1-V^2(x))^{d/2}$,

$$|v_y| = \left|\nabla_p H(y, \varpi(0))\right| = \frac{|\varpi(0)|}{(1-\varpi(0)^2)^{1/2}} = \frac{(1-V^2(y))^{1/2}}{|V(y)|},$$

and, analogously, $|v_x| = (1-V^2(x))^{1/2}/|V(x)|$, we obtain

(6.12)
$$\det\begin{pmatrix} 0 & -v_y^\top \\ v_x & id_\eta Q^+(\tau, y, 0)\end{pmatrix} = \frac{\det\left((c_i^*\, d_p X(\tau, y, \varpi(0))\,(b_j^*)^\top)_{1\leqslant i,j\leqslant d-1}\right)}{|V(x)||V(y)|(1-V^2(x))^{\frac{d-2}{2}}(1-V^2(y))^{\frac{d-2}{2}}}.$$

In order to compare the position space projection, X, of the Hamiltonian flow associated with H and the exponential map at y we observe that

$$X(\tau, y, p) = \exp_y\Big(d_{\mathrm{A}}\big(X(\tau, y, p), y\big)\,(1-V^2(y))^{-1/2}\,p/|p|\Big),$$

for $p \in \mathbb{R}^d$ in some neighborhood of $\varpi(0)$, since $\mathcal{L}(p) := (1-V^2(y))^{-1/2}\,p/|p|$ is normalized with respect to $G(y)$ and the initial momentum p of a Hamiltonian trajectory is collinear with its initial velocity in our case. We set $r(p) := d_{\mathrm{A}}(X(\tau, y, p), y)$. Then it follows that

$$d_p X(\tau, y, p) = \exp'_y\big(r(p)\,\mathcal{L}(p)\big)\big[\mathcal{L}(p)\otimes r'(p)\big] + r(p)\,\exp'_y\big(r(p)\,\mathcal{L}(p)\big)\,\mathcal{L}'(p).$$

By Gauß' lemma we know that $c_i^* \exp_y'(r(\varpi(0))\,\mathcal{L}(\varpi(0)))\,\mathcal{L}(\varpi(0)) = 0$, for $i = 1, \ldots, d-1$, thus

$$c_i^*\, d_p X(\tau, y, \varpi(0))\,(b_j^*)^\top = r(\varpi(0))\, c_i^*\, \exp_y'\big(r(\varpi(0))\,\mathcal{L}(\varpi(0))\big)\,\mathcal{L}'(\varpi(0))\,(b_j^*)^\top,$$

for $i, j = 1, \ldots, d-1$. If $P_{\varpi(0)}^\perp$ denotes the orthogonal projection in $\mathbb{R}^d$ onto the Euclidean orthogonal complement of $\varpi(0)$, we have

$$\mathcal{L}'(\varpi(0))\,(b_j^*)^\top \;=\; (1 - V^2(y))^{-1/2}\,\frac{1}{|\varpi(0)|}\,P_{\varpi(0)}^\perp\,(1 - V^2(y))\,b_j = b_j\,,$$

since $|\varpi(0)| = (1 - V^2(y))^{1/2}$. Using also the identities $r(\varpi(0)) = d_{\mathrm{A}}(x, y)$ and $r(\varpi(0))\,\mathcal{L}(\varpi(0)) = \exp_y^{-1}(x)$ we arrive at

$$\begin{aligned} &\det\big((c_i^*\, d_p X(\tau, y, \varpi(0))\,(b_j^*)^\top)_{1\leqslant i,j\leqslant d-1}\big) \\ &\qquad = d_{\mathrm{A}}(x, y)^{d-1}\,\det\big((c_i^*\,\exp_y'(\exp_y^{-1}(x))\,b_j)_{1\leqslant i,j\leqslant d-1}\big)\,. \end{aligned}$$

Finally, we use $c_d^* \exp_y'(\exp_y^{-1}(x))\,b_d = 1$ to conclude that

$$\det\big((c_i^*\, d_p X(\tau, y, \varpi(0))\,(b_j^*)^\top)_{1\leqslant i,j\leqslant d-1}\big) = d_{\mathrm{A}}(x, y)^{d-1}\,\det\big(\exp_y'(\exp_y^{-1}(x))\big)\,.$$

Inserting this identity into (6.12) we arrive at the assertion. □

Appendix A. Connection to the BMT equation for Thomas precession

In this appendix we consider only the case $d = 3$ and choose the standard representation of the Dirac matrices,

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}\,,\quad j = 1, 2, 3\,,\qquad \alpha_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\,.$$

We know that the matrix-valued term $M(x_\star, y_\star) = U(\tau)\,(-V(y_\star))\,\Lambda^+(i\varpi(0))$ appearing in (1.13) maps $\mathrm{Ran}\Lambda^+(i\varpi(0))$ onto $\mathrm{Ran}\Lambda^+(i\varpi(\tau))$. If we choose appropriate bases of these subspaces then the coefficient matrix of $M(x_\star, y_\star)$ corresponding to these basis vectors is a solution of some spin transport equation which is closely related to the Bargmann-Michel-Telegdi (BMT) equation for the Thomas precession of a classical three-dimensional spin; see, e.g., [1, 8]. In our special case, where no magnetic field is present, the Hamiltonian determining the particle trajectory $\binom{\gamma}{\varpi}$ is H given by (1.9), and $\sqrt{1 - \varpi^2} = -V(\gamma)$, this spin transport equation reads

$$\text{(A.1)}\qquad \frac{d}{dt}\,\mathfrak{s}(t) \;=\; i\,\mathfrak{M}(\gamma(t), \varpi(t))\,\mathfrak{s}(t)\,,\qquad \mathfrak{M}(x, p) \;:=\; \frac{\boldsymbol{\sigma}\cdot\big(E(x)\times p\big)}{-2V(x)[1 - V(x))]}\,,$$

where $E = -\nabla V$ is the electrical field and $\mathfrak{s}(t)$ is a complex (2×2)-matrix. Notice that the usual momentum is replaced by an imaginary momentum in H since we are dealing with some sort of tunneling regime. The BMT equation, or rather its analogue for the Hamiltonian H, is obtained from (A.1) by

choosing some $u \in \mathbb{C}^2$ and computing the differential equation satisfied by the expectation value $\mathbf{s}(t) := \langle\, \mathfrak{s}(t)\, u \,|\, \boldsymbol{\sigma}\, \mathfrak{s}(t)\, u \,\rangle_{\mathbb{C}^2}$ of the vector of Pauli matrices. In our case the BMT equation turns out to be

$$\frac{d}{dt}\,\mathbf{s}(t) \;=\; \frac{\mathbf{s}(t) \times (E(\gamma(t)) \times \varpi(t))}{-V(\gamma(t))[1 - V(\gamma(t))]}\,. \tag{A.2}$$

In order to derive (A.1) and connect it to (1.13) we start with Equation (4.28) for B^0_+, which, on the domain $\mathscr{D}_+$, reads

$$-\,2V(x)\,\Lambda^+(i\nabla\varphi(x))\,(i\partial_t + \boldsymbol{\alpha}\cdot\nabla_x)\,B^0_+(t,x,0) \;=\; 0\,, \qquad (t,x,0) \in \mathscr{D}_+\,. \tag{A.3}$$

For every $x \in \mathbb{R}^3$, the range of $\Lambda^+(i\nabla\varphi(x))$ is spanned by the two mutually orthonormal eigenvectors of $\widehat{D}(x, i\nabla\varphi(x))$ in the (4×2)-matrix

$$W(x) \;:=\; \frac{1}{\sqrt{-2V(x)[1 - V(x)]}} \begin{pmatrix} [1 - V(x)]\,\mathbb{1} \\ \boldsymbol{\sigma}\cdot i\nabla\varphi(x) \end{pmatrix}, \tag{A.4}$$

so that $\Lambda^+(i\nabla\varphi(x)) = W(x)\,W(x)^\top$. Next, we write $B^0_+(t,x,0) = W(x)\,C(t,x)$, for some (2×4)-matrix $C(t,x)$. This is possible since B^0_+ fulfills $(\boldsymbol{T_0})$. Then (A.3) implies $W^\top\,(i\partial_t + \boldsymbol{\alpha}\cdot\nabla_x)\,W\,C = 0$ and we observe as in [1] that the operator $-iW^\top\,(i\partial_t + \boldsymbol{\alpha}\cdot\nabla_x)\,W$ equals

$$\partial_t + F\cdot\nabla_x + \frac{1}{2}\,\mathrm{div}F - i\,\mathfrak{M}(x,\nabla\varphi)\,, \quad F(x) := \nabla_p H(x,\nabla\varphi(x)) = V(x)^{-1}\nabla\varphi(x)\,.$$

Substituting γ for x and using $\dot\gamma = F(\gamma)$, we deduce that

$$\frac{d}{dt}\,C(t,\gamma(t)) \;=\; -\frac{1}{2}\,\mathrm{div}F(\gamma(t))\,C(t,\gamma(t)) + i\,\mathfrak{M}(\gamma(t),\varpi(t))\,C(t,\gamma(t))\,.$$

Notice that, unlike (4.35), the previous equation involves the complete divergence of F. Using the ansatz $C(t,\gamma(t)) = \varrho(t)\,\mathfrak{s}(t)\,W(y_\star)^\top$, where ϱ is scalar, and $\mathfrak{s}$ solves (A.1) with $\mathfrak{s}(0) = \mathbb{1}_2$, we thus find by means of Liouville's formula and the initial condition $C(0,y_\star) = W(y_\star)^\top$ that $\varrho(t) = \det[d_y Q_+(t,y_\star,0)]^{-1/2}$, $t\in[0,\tau]$. We arrive at the following formula for B^0_+ alternative to (4.34),

$$B^0_+(\tau,x_\star,0) \;=\; \det[d_y Q_+(t,y_\star,0)]^{-1/2}\,W(x_\star)\,\mathfrak{s}(\tau)\,W(y_\star)^\top.$$

Using the previous formula instead of (4.34) in the proof of Proposition 6.1 we obtain the following asymptotics for the Green kernel,

$$\begin{aligned} &D^{-1}_{h,V}(x_\star,y_\star) \\ &= \frac{(1 - V^2(x_\star))^{\frac14}(1 - V^2(y_\star))^{\frac14}\big(V(x_\star)V(y_\star)\big)^{\frac12}}{h^3\,\det\big[\,\exp'_{y_\star}(\exp^{-1}_{y_\star}(x_\star))\big]^{1/2}}\cdot\frac{(1+\mathcal{O}(h))\,e^{-d_{\mathrm{A}}(x_\star,y_\star)/h}}{2\pi\,d_{\mathrm{A}}(x_\star,y_\star)/h} \\ &\quad\cdot W(x_\star)\,\mathfrak{s}(\tau)\,W(y_\star)^\top, \qquad \mathfrak{s}\text{ solves (A.1)},\ \mathfrak{s}(0) = \mathbb{1}_2\,. \end{aligned}$$

Here W is given by (A.4) where $\nabla\varphi(x_\star) = \varpi(\tau)$, $\nabla\varphi(y_\star) = \varpi(0)$. We remark without proof that the scalar term in the second line is the asymptotic expansion of the Green kernel of the Weyl quantization of $\sqrt{1+\xi^2} + V$ in three dimensions. This can be inferred by means of a procedure similar to the one carried through in the present paper. Finally, we remark that $\mathfrak{s}(\tau)$ can be represented by means of the polar coordinates of a suitable solution of (A.2) and additional dynamical and geometric phases; see [1, §4].

Appendix B. Proof of Lemma 3.3

In order to give a self-contained construction of the phase functions $\psi_\pm$ we present the proof of Lemma 3.3 in this appendix. The proofs below are variants of those in [7] where the symbol is assumed to be homogeneous of degree one. We recall the definition of $\mathfrak{S}$ in (3.8) and start with the following lemma which corresponds to [7, Lemma 1.7].

Lemma B.1. *For every compact subset $K \subset \Omega$, we find some $C_K \in (0,\infty)$ such that, for all $(s,\rho) = (s,x,\xi) \in \mathbb{C} \times K$,*

$$(\widehat{\mathscr{K}_{a_\pm}}\,\mathfrak{S})(s,\rho) \,\geqslant\, -\frac{3}{4}\,\Im a_\pm(\Re\rho) - C_K\,|\Im\rho|^3. \tag{B.1}$$

Proof. Let a be a_+ or a_-. Since $\widehat{\mathscr{K}_a}$ is a real differential operator we have $[\widehat{\mathscr{K}_a}, \Re] = 0$, whence

$$\begin{aligned}\widehat{\mathscr{K}_a}\big(s + \langle\, x\,|\,\Re\xi\,\rangle\big) \;&=\; a - \langle\, \nabla_\xi a\,|\,\xi\,\rangle + \langle\, \nabla_\xi a\,|\,\Re\xi\,\rangle - \langle\, x\,|\,\Re\nabla_x a\,\rangle \\ &=\; a - \langle\, \nabla_\xi a\,|\,i\Im\xi\,\rangle - \langle\, i\Im x\,|\,\nabla_x a\,\rangle - \Re\langle\, \overline{x}\,|\,\nabla_x a\,\rangle\end{aligned}$$

on Ω. Taking the imaginary part and using that $[\widehat{\mathscr{K}_a}, \Im] = 0$ we obtain

$$-\widehat{\mathscr{K}_a}\,\mathfrak{S} \;=\; \Im\big(a - \langle\, \nabla_\rho a\,|\,i\Im\rho\,\rangle\big)\,.$$

Taylor expanding both a and $\nabla_\rho a$ at $\Re\rho$ using $\partial^\alpha_{(\Re\rho,\Im\rho)}\nabla_{\overline{\rho}} a(\Re\rho) = 0$, $\alpha \in \mathbb{N}_0^{4d}$, we infer that

$$a - \langle\, \nabla_\rho a\,|\,i\Im\rho\,\rangle \;=\; a(\Re\rho) \,+\, \frac{1}{2}\,\langle\, \Im\rho\,|\,a''_{\rho\rho}(\Re\rho)\,\Im\rho\,\rangle + \mathcal{O}\big(|\Im\rho|^3\big)\,. \tag{B.2}$$

Furthermore, a Taylor expansion of $\Im a(\Re\rho \pm t\Im\rho)$ with $t > 0$ yields

$$\begin{aligned}&\frac{1}{t^2}\,\Im a(\Re\rho) + \frac{1}{2}\,\langle\, \Im\rho\,|\,\Im a''_{\rho\rho}(\Re\rho)\,\Im\rho\,\rangle \\ &\quad= \frac{1}{2t^2}\,\big(\Im a(\Re\rho + t\Im\rho) + \Im a(\Re\rho - t\Im\rho)\big) + t\,\mathcal{O}\big(|\Im\rho|^3\big)\,.\end{aligned} \tag{B.3}$$

The $\mathcal{O}$-symbols in (B.2) and (B.3) are uniform when ρ varies in a compact subset of Ω. Choosing $t = 2$ and using that $\Im a \leqslant 0$ we thus arrive at the assertion. □

Proof of Lemma 3.3. We drop all sub- and superscripts $\pm$ in this proof. Taylor expanding the right side of $\frac{d}{dt}\,\Im\kappa_t = \Im\widehat{\mathscr{H}_a}(\kappa_t)$ at $\Re\kappa_t$ and using Duhamel's formula we obtain, exactly as in [7, pp. 351], the estimate

$$|\Im\kappa_u| \leqslant \mathcal{O}(1)\left(|\Im\kappa_t| + \int_u^t |\Im a'(\Re\kappa_r)|\,dr\right). \tag{B.4}$$

It holds for all ρ in a compact, complex neighborhood of (y_0,η_0) and $u,t\in[0,\tau+\varepsilon_1]$, for some $\varepsilon_1>0$. Since, for ρ in a compact set, the curves $[0,\tau+\varepsilon_1]\ni t\mapsto\Re\kappa_t(\rho)$ stay in a compact set, we may apply the standard estimate for positive functions to (B.4), which together with Hölder's inequality gives

$$|\Im\kappa_r| \leqslant \mathcal{O}(1)\left(|\Im\kappa_t| + (I_u^t)^{1/2}\right), \qquad I_u^t(\rho) := \int_u^t -\Im a(\Re\kappa_v(\rho))\,dv, \tag{B.5}$$

for $0\leqslant u\leqslant r\leqslant t\leqslant\tau+\varepsilon_1$. Next, we integrate the estimate (B.1) for $\frac{d}{dt}\mathfrak{S}(\varsigma_t,\kappa_t) = \widehat{\mathscr{K}_a}(\mathfrak{S})(\varsigma_t,\kappa_t)$ from u to t, use (B.5) to bound $|\Im\kappa_r|$, $r\in[u,t]$, and arrive at

$$\mathfrak{S}(\varsigma_t,\kappa_t) \geqslant \mathfrak{S}(\varsigma_u,\kappa_u) + \frac{3}{4}\,I_u^t - \mathcal{O}(1)\,(t-u)\left(|\Im\kappa_t|^3 + (I_u^t)^{3/2}\right). \tag{B.6}$$

In the case $u=0$ we further observe by means of (3.7) and (B.5) that

$$\mathfrak{S}(\varsigma_0,\kappa_0) \geqslant -\mathcal{O}(1)\,|\Im\rho|^3 \geqslant -\mathcal{O}(1)\left(|\Im\kappa_t|^3 + (I_0^t)^{3/2}\right). \tag{B.7}$$

If $\tau>0$ in the plus-case we use the assumption $\Im a_+(y_0,\eta_0)=0$ and Lemma 3.2 to deduce that $\Im a_+(\kappa_t^+(y_0,\eta_0))=0$, $t\in[0,\tau]$, thus $I_u^t(y_0,\eta_0)=0$, $0\leqslant u\leqslant t\leqslant\tau$. Back in the general case we conclude that, for $0\leqslant u\leqslant t\leqslant\tau+\varepsilon_1$, we can ensure that the integrals $I_u^t(\rho)$ are as small as we please by assuming that ρ is contained in a sufficiently small neighborhood of (y_0,η_0), and that $\varepsilon_1>0$ is sufficiently small. Then (B.6) and (B.7) give (3.9) and the estimate

$$\mathfrak{S}(\varsigma_t,\kappa_t) \geqslant \mathfrak{S}(\varsigma_u,\kappa_u) + \frac{1}{2}\,I_u^t - \mathcal{O}(1)\,(t-u)\,|\Im\kappa_t|^3, \tag{B.8}$$

for $0\leqslant u\leqslant t\leqslant\tau+\varepsilon_1$. Squaring (B.5) with $r=u$, dividing by some suitable constant, and adding the result to (B.8) we obtain

$$\mathfrak{S}(\varsigma_t,\kappa_t) + \frac{1}{4}|\Im\kappa_t|^2 \geqslant \mathfrak{S}(\varsigma_u,\kappa_u) + \frac{2}{C}|\Im\kappa_u|^2 + \left(\frac{1}{2}-\frac{1}{4}\right)I_u^t - \mathcal{O}(1)(t-u)|\Im\kappa_t|^3,$$

where we can assume that the constant (coming from (B.5)) satisfies $C\geqslant 4$. By possibly restricting the neighborhood of (y_0,η_0) further to ensure that $\max_{t\in[0,\tau+\varepsilon_1]}|\Im\kappa_t|$ is sufficiently small, we infer from the previous estimate that

$$\mathfrak{S}(\varsigma_t,\kappa_t) + |\Im\kappa_t|^2 \geqslant \mathfrak{S}(\varsigma_u,\kappa_u) + \frac{1}{C}\,|\Im\kappa_u|^2 + \frac{1}{4}\,I_u^t, \tag{B.9}$$

for $0\leqslant u\leqslant t\leqslant\tau+\varepsilon_1$. On account of (3.9), where the integral is non-negative, we may again restrict the neighborhood of (y_0,η_0) so that $|\Im\kappa_u(\rho)|$ is sufficiently

small to ensure that $(1-\frac{1}{C})\mathfrak{S}(\varsigma_u,\kappa_u)+\frac{1}{C}|\Im\kappa_u|^2\geqslant 0$, for all $u\in[0,\tau+\varepsilon_1]$. Subtracting the latter term from the right hand side of (B.9) and using $C\geqslant 4$ we arrive at

$$\big(\mathfrak{S}(\varsigma_t,\kappa_t)+|\Im\kappa_t|^2\big)\geqslant\frac{1}{C}\,\big(\mathfrak{S}(\varsigma_u,\kappa_u)+|\Im\kappa_u|^2\big)+\frac{1}{C}\,I_u^t\,,$$

which is (3.10). Subtracting $\frac{1}{C}(\mathfrak{S}(\varsigma_r,\kappa_r)+\frac{1}{2}\,|\Im\kappa_r|^2)\geqslant 0$ from the right hand side of (3.10), where the integral is non-negative, we finally obtain (3.11). □

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